Fiber Bundles in Gravitational Theory

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1. Introduction

In 1954, Yang and Mills introduced a new idea into theoretical physics, which later came to dominate the physicist's view of the fundamental structure of the physical forces of nature. The Yang-Mills theory is the 'gauge theory of a non-Abelian symmetry group', and is essentially a generalization of Maxwell's theory of electromagnetism, which is the gauge theory of a one-parameter Abelian group.

The theory of Yang and Mills dealt specifically with the isospin symmetry of nuclear forces. The gauging of the isospin group SU(2) leads to a triplet of isospin-1 mesons as analogues of the photon; the non-Abelian nature of the group gives rise to nonlinearity of their field equations. The eminently successful Salam-Weinberg unification of the weak and electromagnetic forces (1973) exploited the Yang-Mills ideas. The observed distinction between the weak forces and the electromagnetic forces, including the masses of the $W_{\pm}$ and the Z, come from a spontaneous symmetry breaking mechanism (Higgs mechanism). Quantum chromodynamics (QCD) is, similarly, a gauge theory for the strong interactions, in which the Yang-Mills particles are the 'gluons' that mediate the forces between quarks. Attempts to unify the electromagnetic, weak and strong forces by further exploiting the Yang-Mills idea are the 'grand unified theories' (GUTs). The hope is to find a group that contains the Salam-Weinberg group and the unitary symmetry group of QCD in a non-trivial way, to gauge the group, and to introduce appropriate Higgs mechanisms to break the symmetry so as to obtain the observed behaviour of the three kinds of fundamental interaction.

In all these developments, the gravitational forces are 'conspicuous by their absence'. From the outset, Einstein's general relativity has stood alone, isolated from the developments that have taken place in our understanding of the other fundamental forces. This immense conceptual gap was the source of Einstein's opposition to the developments that took place in physics as a result of the advent of quantum theory, an opposition summarized in his famous statement "God does not play dice." At present the gap appears not quite so unbridgeable (though the problem of the unification of all the forces of nature, including gravitation, is still formidable). In fact, Einstein's gravitational theory and various modifications and extensions of it, can be understood as 'gauge theories' in the Yang-Mills sense. We shall not discuss the physics of these theories; our aim here is only to throw some light on the geometrical concepts that allow gravitational theories to be viewed as gauge theories. More details will be found in the references.

2. Gauging a Non-Abelian Group

Let ψ be a set of physical fields of a Lagrangian theory invariant under a group of linear transformations

$$\psi \to S\psi. \tag{1}$$

When S is made space-time dependent, invariance is maintained by replacing derivatives of ψ by a generalized derivative

$$D_i\psi = \partial_i\psi + \Gamma_i\psi, \tag{2}$$

where Γ_i is a linear combination of the generators G_a of the group,

$$\Gamma_i = \Gamma_i{}^a G_a, \tag{3}$$

provided the 'gauge potentials' $\Gamma_i{}^a$ have the transformation law

$$\Gamma_i \to S\Gamma_i S^{-1} - (\partial_i S)S^{-1}. \tag{4}$$

The 'gauge fields' $F_{ij}{}^a$ are defined by

$$[D_i, D_j] = F_{ij} = F_{ij}{}^a G_a = \partial_i\Gamma_j - \partial_j\Gamma_i + [\Gamma_i, \Gamma_j]. \tag{5}$$

They transform homogeneously:

$$F_{ij} \to SF_{ij}S^{-1}. \tag{6}$$

This is the basic Yang-Mills idea. We have a generalization of the electromagnetic potential and the electromagnetic field. Under an infinitesimal gauge transformation and an infinitesimal coordinate transformation,

$$S = I + \epsilon, \quad \epsilon = \epsilon^a G_a, \quad x^i \to x^i - \xi^i \tag{7}$$

we have

$$\delta\psi = \xi^j\partial_j\psi + \epsilon\psi, \tag{8}$$

$$\delta\Gamma_i = \xi^j\partial_j\Gamma_i + (\partial_i\xi^j)\Gamma_j - D_i\epsilon, \quad D_i\epsilon = \partial_i\epsilon + [\Gamma_i, \epsilon]. \tag{9}$$

Changing the parameters to $\lambda = \xi^i\Gamma_i - \epsilon$, we have the following neat 'manifestly invariant' forms for the infinitesimal changes:

$$\delta\psi = \xi^j D_j\psi - \lambda\psi, \tag{10}$$

$$\delta\Gamma_i = \xi^j F_{ji} + D_j\lambda. \tag{11}$$

3. Lie Groups

A Lie group G is a group whose elements constitute a differentiable manifold. An element of G can be regarded as a transformation on the manifold, or as a point of the manifold. We shall write g to denote an element of G when the former aspect is emphasized, and we shall write z when we wish to emphasize the latter aspect. Associated with any element g, there is a transformation on the manifold G, called *left translation*:

$$z \to L_g z = gz. \tag{12}$$

Similarly, right translation is defined by

$$z \to R_g z = zg. \tag{13}$$

A left-invariant vector field on G is a vector field that is unchanged by any left translation. A left-invariant field generates a one-parameter group of right translations, and vice-versa. The commutation of two left-invariant vector fields is a left-invariant vector field. So the left-invariant vector fields form an algebra under commutation, called the *Lie algebra* of G. A basis for the Lie algebra is a set $\{R_A\}$ of left-invariant vector fields. It constitutes a *vielbein* on the manifold G. [*Vielbein*: German for 'many legs', a generalization of the terminology *vierbein* ('four legs') meaning 'tetrad'.] We write $A, B, \ldots$ for the vielbein labels and $M, N, \ldots$ for coordinate-based indices. Denote the elements of the matrix of components of the 'left-vielbein' $\{R_A\}$ by $R_A{}^M$. The elements of the inverse matrix can then be denoted by $R_M{}^A$; they are the components of a set $\{R^A\}$ of 'one-forms' (i.e., covariant as opposed to contravariant vectors), constituting the basis 'dual' to $\{R_A\}$. The *structure constants* of the group G are given by

$$[R_A, R_B] = C_{AB}{}^C R_C. \tag{14}$$

Let S be any matrix representation of the Lie group G; i.e., $S(z)$ is a matrix field on the manifold G satisfying $S(g)S(z) = S(gz)$, $S(z^{-1}) = S^{-1}(z)$. The matrix generators G_A for the representation S are

$$G_A = S^{-1}R_A(S). \tag{15}$$

They can be shown to satisfy

$$[G_A, G_B] = C^C_{AB}G_C. \tag{16}$$

An ordinary one-form maps a vector to a scalar. A 'Lie algebra-valued' one-form maps a vector to an element of the Lie algebra, i.e., to a left-invariant vector field. The *Maurer-Cartan form* θ is the one-form that maps a vector X at a point z of G to the unique left-invariant field that takes the value X at z. In a representation S, θ is represented by a matrix-valued one-form

$$\theta = \theta^A G_A, \tag{17}$$

where the coefficients θ^A are ordinary one-forms. Since $\theta^A G_A$ maps R_B to its representative matrix G_B, we have $\theta^A(R_B) = \delta^A{}_B$, so $\theta^A = R^A$. Therefore

$$\theta = G_A R^A. \tag{18}$$

Observe also that the matrix-valued one-form $S^{-1}\,dS$ satisfies

$$(S^{-1}\,dS)R_A = S^{-1} R_A(S) = G_A,$$

and hence

$$\theta = S^{-1}\,dS \tag{19}$$

in any representation S. If G is a matrix group, we can use the self representation and write simply

$$\theta = z^{-1}\,dz. \tag{20}$$

Then, from $dz = z\theta$ and $d\,dz = 0$ we get the *Maurer-Cartan equation*

$$d\theta + \theta \wedge \theta = 0. \tag{21}$$

In terms of components, this is

$$\partial_M R_N{}^A - \partial_N R_M{}^A + R_M{}^B R_N{}^C C_{BC}{}^A = 0, \tag{22}$$

which is equivalent to

$$R_A{}^M \partial_M R_B{}^N - R_B{}^M \partial_M R_A{}^N = C_{AB}{}^C R_C{}^N. \tag{23}$$

That is, the Maurer-Cartan equation and the commutation relations (14) are equivalent.

4. Fiber Bundles

The theory of fiber bundles was developed by mathematicians as a branch of pure mathematics. Exciting developments began in the 1960s with the realization that the mathematicians' 'fiber bundles' and the physicists' 'gauge theories' were essentially identical. The mathematicians' preoccupation with the global topological properties of the geometrical structures known as fiber bundles then provided physicists with methods and concepts that released the study of gauge theories from its preoccupation with local concepts (formulated in terms of differential equations).

A fiber bundle is constructed from a *bundle space* P and a Lie group G, the *structural group*, which acts on P without fixed points. The orbits of the action of G are the fibers, which are subspaces of P. There is one fiber F_z through each point z of P. The fibers are required to be all homeomorphic to a *fiber space* F so that the action of G on P corresponds to an action of G on F. The set of all fibers is homeomorphic to a space M, the *base space*, and a projection operator π maps P to M, each point z being mapped to a unique point $x = \pi z \in M$ so that all the points of a fiber are mapped to the same point of M. A point $x \in M$ specifies a unique fiber $F_x = \pi^{-1}x$ in P.

With hindsight, one can see that the first use of a fiber bundle in physics was in fact the Kaluza-Klein theory. The group G is the electromagnetic gauge group, the bundle space is five-dimensional and the fibers one-dimensional. The four-dimensional base-space is space-time.

A *principal fiber bundle* $P(M,G)$ is a fiber bundle for which the fiber space F is the manifold of the structural group G. We denote the action of an element g of G on P, by the notation

$$z \to zg = R_g z. \tag{24}$$

An equivariant field Ψ on P is a field belonging to a linear representation S of G with the transformation law

$$R^*_{g^{-1}}\Psi = S(g)\Psi. \tag{25}$$

This can be written as

$$\Psi(zg^{-1}) = S(g)\Psi(z) \tag{26}$$

so that an equivariant field can be seen to be determined on the whole of a fiber F_z if its value at any point z of the fiber is given.

Consider the tangent space to P, at a point z. The tangent space to the fiber F_z, at z, is a subspace. One can choose a space H_z so that any vector z can be resolved into a component in F_z (called the 'vertical' component) and a component in H_z (called the 'horizontal' component):

$$X = X_v + X_h \tag{27}$$

(see Fig. 1). If a 'horizontal space' H_z is chosen at every point of P in such a way that the whole set of horizontal spaces is invariant under the action of G, then the set of horizontal spaces is called an 'Ehresmann connection' on the principal fiber bundle.

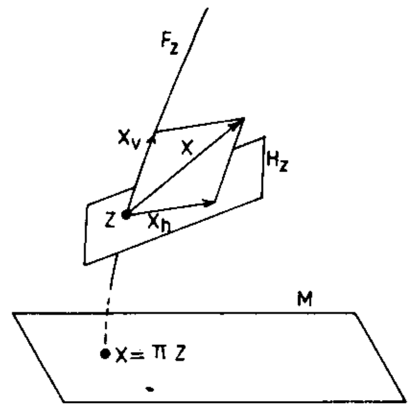

Fig. 1

Given a vector Y at a point x of the base space M, one can define a unique vector $\bar{Y}$ at any point z of the fiber F_x, that is horizontal and is mapped to Y by the projection π. This is the 'horizontal lift' of Y at z. Moreover, a curve in M can be 'lifted' to give 'horizontal' curves in P, and one can then proceed to introduce the idea of a 'parallel transport' of a fiber, or of an equivariant field, along a curve in M. Clearly, a horizontal lift of a closed curve in M is not, in general, a closed curve in P (Fig. 2) and the concept of a connection in the Ehresmann sense leads to a corresponding concept of 'curvature'.

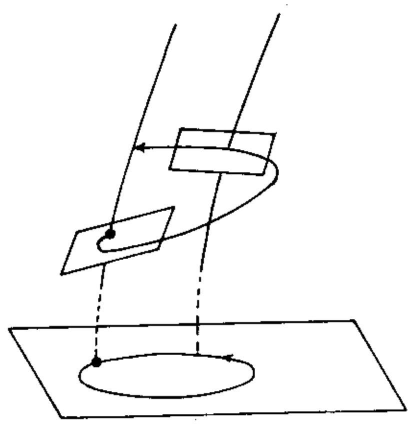

Fig. 2

An alternative definition of connection, equivalent to the Ehresmann definition, is as follows. A *connection* on a principal fiber bundle $P(M,G)$ is a Lie algebra-valued one-form ω that is equivariant,

$$R^*_{g^{-1}}\omega = g\omega g^{-1} \tag{28}$$

and that maps any vertical vector at a point z to the corresponding left-invariant field on F:

$$\omega(X_v) = \theta(X_v). \tag{29}$$

Given such an ω, Ehresmann's horizontal spaces can be constructed from those vectors X_h that satisfy $\omega(X_h) = 0$. The *curvature* associated with a connection ω is defined to be the Lie algebra-valued two-form

$$\Theta = 2(d\omega + \omega \wedge \omega). \tag{30}$$

A *section* on a fiber bundle is a mapping $\sigma: M \rightarrow P$ satisfying $\sigma\pi = 1$. That is, σ associates with each point x on M a unique point $\sigma(x)$ in the fiber F_x (Fig. 3). One can then define 'pull-backs' of fields on P:

$$\psi = \sigma^*\Psi, \quad \Gamma = \sigma^*\omega, \quad F = \sigma^*\Theta \tag{31}$$

are fields on M. Under the action of G on P, they transform according to Eq. (1), Eq. (4) and Eq. (6). Hence *the theory of a principal fiber bundle provides a geometrical realization of a gauge theory.*

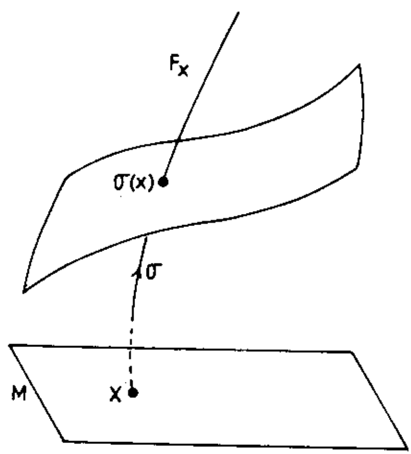

Fig. 3

A gauge transformation can be interpreted *actively*, as a mapping on the bundle space, or *passively*, as a change of section. The section has no physical content, it is simply a part of the reference system, enabling equivariant fields on P to be described in terms of fields on space-time M; all sections are equivalent so far as the physics is concerned.

The gauge transformations do not affect the points of space-time: the fibers are acted upon but not moved by an active gauge transformation. So the fiber bundle theory we have described is of use only for the gauge theory of an *internal* symmetry group.

Now, as was shown by Kibble for the Poincaré group, the Yang-Mills idea of gauging a symmetry group can be applied also to space-time groups, as well as to internal symmetries. Indeed, in the case of the Poincaré group, the Yang-Mills trick led to a theory similar to Einstein theory, but with non-vanishing torsion (the ECKS theory). To incorporate this kind of extension of the Yang-Mills idea into fiber bundle language, one needs to consider transformations on a bundle space that shift the fibers around. A very elegant way of doing this was introduced by Ne'eman and Regge. The base space in their approach is a coset space. The version of coset bundle theory that we describe below was developed by Lord and Goswami.

5. Coset Bundles

Let G be a Lie group and H a Lie subgroup. Let H act on G by right translation:

$$z \to R_h z = zh. \tag{32}$$

We then have a principal fiber bundle $G(G/H, H)$. The bundle space is the manifold G. The structural group is H (acting on the right) and the fibers are the cosets zH. The base space is the coset space G/H, which will be interpreted as space-time. G may contain internal symmetries as well as space-time symmetry (such as the Poincaré group, the de Sitter group or the conformal group).

Consider the *left* action of the *whole* of G on the coset bundle:

$$z \to L_g z = gz. \tag{33}$$

The points of the base space are *not* invariant under this action. Writing $x = \pi z$, the effect on the base space is

$$x \to x' = \pi g \pi^{-1} x. \tag{34}$$

If a section σ is introduced, the action can be conceived as consisting of a 'space-time dependent' action of the structural group H, and an action [Eq. (34)] on space-time:

$$g\sigma(x) = \sigma(x')h(x, g). \tag{35}$$

We now define *gauge transformations* to be the most general transformations on the space $G(G/H, H)$ that preserve the fiber bundle structure. That is, a gauge transformation $z \to f(z)$ is a mapping that commutes with the right action of the structural group H:

$$f(z)h = f(zh). \tag{36}$$

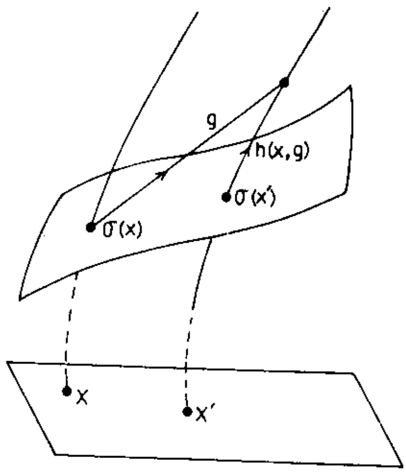

Fig. 4

Defining $g(z) = f(z)z^{-1}$, we find $g(zh) = g(z)$, so that $g(z)$ is constant over each fiber, and so we can write $g(z) = g(x)$. Then

$$f(z) = g(x)z. \tag{37}$$

Thus, a gauge transformation, as we have defined it, is like a left-translation, except that the group element g is *space-time dependent.*

A *generalized connection* ω on the coset bundle is defined to be a Lie algebra-valued one-form, by which we mean *the Lie algebra of* G, *not just of the structural group* H. Moreover, we require ω to satisfy

1. $R^*_{h^{-1}}\omega = h\omega h^{-1}$ (equivariance);
2. $\omega(X) = 0$ if and only if $X = 0$ (nonsingular);
3. $\omega(X_v) = \theta(X_v)$ for any vertical vector.

The generalized curvature associated with ω is defined to be

$$\Theta = 2(d\omega + \omega \wedge \omega). \tag{38}$$

Let the coordinates of a point z of G be denoted by z^M. Adapt the coordinate system to the structure of the bundle by writing $z^M = (x^i, \chi^m)$ where x^i are coordinates on the base-space G/H and χ^m are coordinates on the fiber H. Corresponding to this splitting $M = (i, m)$ of the coordinate-based indices, we can introduce a splitting $A = (\alpha, a)$ of the vielbein indices, where a labels vertical vectors of the vielbein and α labels the rest. We can choose the left vielbein (R_A) so that the (R_a) are vertical. Then $R_a{}^i = 0$ and hence $R_m{}^\alpha = 0$. The third condition on ω then means that the components $E_M{}^A$

of ω have the form

$$E_M{}^A = \begin{pmatrix} E_i{}^\alpha & E_i{}^a \\ 0 & R_m{}^a \end{pmatrix} \tag{39}$$

The equivariance of ω then implies that the only nonvanishing components $\Theta_{MN}{}^A$ of the curvature are $\Theta_{ij}{}^A$.

Let σ be a section. The components of $\sigma(x)$ have the form

$$\sigma^M(x) = \big(x^i, \sigma^m(x)\big). \tag{40}$$

Employing σ to 'pull back' equivariant fields, we define

$$\psi = \sigma^*\Psi = \Psi(\sigma). \tag{41}$$

$\Gamma = \sigma^*\omega$. That is,

$$\left.\begin{aligned} \Gamma_i{}^A &= \sigma^M{}_{.i} E_M{}^A(\sigma), \\ \text{or, more explicitly,}\quad \Gamma_i{}^\alpha &= E_i{}^\alpha(\sigma) = e_i{}^\alpha, \\ \Gamma_i{}^a &= E_i{}^a(\sigma) + \sigma^m{}_{.i} R_m{}^a(\sigma). \end{aligned}\right\} \tag{42}$$

$F = \sigma^*\Theta$. That is,

$$F_{ij}{}^A = \sigma^M{}_{.i}\sigma^N{}_{.j}\Theta_{MN}{}^A(\sigma) = \Theta_{ij}{}^A(\sigma). \tag{43}$$

We find that

$$F_{ij} = \partial_i\Gamma_j - \partial_j\Gamma_i + [\Gamma_i, \Gamma_j], \tag{44}$$

i.e.,

$$F_{ij}{}^A = \partial_i\Gamma_j{}^A - \partial_j\Gamma_i{}^A + \Gamma_i{}^B\Gamma_j{}^C C_{BC}{}^A. \tag{45}$$

We can compute the action of an infinitesimal gauge transformation $z^M \to z^M - \Lambda^M$ on these space-time fields. The gauge transformation is determined by the parameters

$$\lambda^A(x) = \sigma^* E^A\Lambda = \Lambda^A(\sigma). \tag{46}$$

We find

$$\delta\psi = \sigma^*\Lambda\Psi = \lambda^A Q_A\psi, \tag{47}$$

where

$$Q_A\psi = \sigma^* E_A\Psi = \sigma^* E_A{}^M\Psi_M.$$

But

$$\partial_i\psi = \sigma^M{}_{.i}\Psi_{.M}(\sigma) = \Psi_{.i}(\sigma) + \sigma^m_{.i}\Psi_{.m}(\sigma),$$

and the infinitesimal form of the equivariance condition on Ψ is $\Psi_{.m} = -R_m{}^a G_a\Psi$. Substituting these expressions, we get

$$Q_A\psi = E_A{}^i(\sigma)(\partial_i\psi + \Gamma_i{}^a G_a\psi) - \delta_A{}^a G_a\psi,$$

i.e.,

$$Q_\alpha\psi = e_\alpha{}^i D_i\psi, \quad D_i\psi = \partial_i\psi + \Gamma_i{}^a G_a\psi, \quad \text{and} \quad Q_a\psi = -G_a\psi.$$

Defining $\xi^i = \lambda^\alpha e_\alpha{}^i$, we obtain the transformation law

$$\delta\psi = \xi^i D_i\psi - \lambda^a G_a\psi. \tag{48}$$

$\delta\Gamma_i^A = \sigma^M{}_{,i}\delta E_M{}^A(\sigma)$. Now $E_M{}^A$ is a covariant vector on the manifold G, so

$$\begin{aligned}\delta E_M{}^A &= \Lambda^N\partial_N E_M{}^A + (\partial_M\Lambda^N)E_N{}^A \\ &= \partial_M\Lambda^A + \Lambda^N(\partial_N E_M{}^A - \partial_M E_N{}^A).\end{aligned}$$

But

$$\Theta_{MN}{}^A = \partial_M E_N{}^A - \partial_N E_M{}^A + E_M{}^B E_N{}^C C_{BC}{}^A,$$

so (using the vielbein components $E_M{}^A$ for converting indices),

$$\delta E_M{}^A = \partial_M\Lambda^A + \Lambda^B(C_{MB}{}^A - \Theta_{MB}{}^A).$$

Therefore,

$$\begin{aligned}\delta\Gamma_i{}^A &= \sigma^M{}_{,i}(\partial_M\lambda^A + \lambda^B E_M{}^C(\sigma)C_{CB}{}^A - \lambda^B E_B{}^N(\sigma)\Theta_{MN}{}^A(\sigma)) \\ &= (\partial_i\lambda^A + \lambda^B\Gamma_i{}^C C_{CB}{}^A) - \xi^j F_{ij}{}^A.\end{aligned}$$

We obtain the transformation law of the potentials in the form

$$\left.\begin{aligned}\delta\Gamma_i{}^A &= \xi^j F_{ji}{}^A + \mathcal{D}_i\lambda^A, \\ \mathcal{D}_i\lambda^A &= \partial_i\lambda^A - \lambda^B\Gamma_i{}^C C_{BC}{}^A.\end{aligned}\right\} \tag{49}$$

The fiber bundle theory has provided us with the appropriate generalizations of the transformation laws given by Eq. (1) and Eq. (4), for the case where G contains a space-time symmetry.

6. Poincaré Gauge Theory

Finally, we shall illustrate the transformation laws we have found, by applying them to a particular example. Let G be the Poincaré group. Denote the generators of Lorentz rotations by $G_{\alpha\beta}$ $(= -G_{\beta\alpha})$ and translation generators by G_α. H is the Lorentz group. G/H is space-time. The commutation relations are:

$$\left.\begin{aligned}&[G_\alpha, G_\beta] = 0, \\ &[G_{\alpha\beta}, G_\gamma] = \eta_{\alpha\gamma}G_\beta - \eta_{\beta\gamma}G_\alpha, \\ &[G_{\alpha\beta}, G_{\gamma\delta}] = \eta_{\alpha\gamma}G_{\beta\delta} - \eta_{\beta\gamma}G_{\alpha\delta} + \eta_{\beta\delta}G_{\alpha\gamma} - \eta_{\alpha\delta}G_{\beta\gamma}\end{aligned}\right\} \tag{50}$$

Writing

$$\Gamma_i = e_i{}^\alpha G_\alpha + \frac{1}{2}\Gamma_i{}^{\alpha\beta}G_{\alpha\beta}, \tag{51}$$

$$F_{ij} = \partial_i\Gamma_j - \partial_j\Gamma_i + [\Gamma_i, \Gamma_j] = F_{ij}{}^\alpha G_\alpha + \frac{1}{2}F_{ij}{}^{\alpha\beta}G_{\alpha\beta}, \tag{52}$$

we find

$$F_{ij}{}^{\beta} = \partial_i e_j{}^{\beta} - \partial_j e_i{}^{\beta} - e_i{}^{\gamma}\Gamma_{j\gamma}{}^{\beta} + e_j{}^{\gamma}\Gamma_{i\gamma}^{\beta}, \tag{53}$$

$$F_{ij\alpha}{}^{\beta} = \partial_i \Gamma_{j\alpha}{}^{\beta} - \partial_j \Gamma_{i\alpha}{}^{\beta} - \Gamma_{i\alpha}{}^{\gamma}\Gamma_{j\gamma}{}^{\beta} + \Gamma_{j\alpha}{}^{\gamma}\Gamma_{i\gamma}{}^{\beta}, \tag{54}$$

(where $\eta_{\alpha\beta}$ has been used as a raising-lowering operator). The translational components $e_i{}^{\alpha}$ of Γ_i define a tetrad, and we can construct a metric and a set of connection coefficients on space-time:

$$g_{ij} = e_i{}^{\alpha} e_j{}^{\beta} \eta_{\alpha\beta}, \tag{55}$$

$$\Gamma_{ij}{}^{k} = (\partial_i e_j{}^{\alpha} + e_j{}^{\beta}\Gamma_{i\beta}{}^{\alpha}) e_{\alpha}{}^{k}. \tag{56}$$

Then the 'translational gauge potentials' $F_{ij}{}^{k}$ turn out to be the torsion, and the 'rotational gauge potentials' $F_{ijk}{}^{l}$ turn out to be the curvature, associated with these connection coefficients. The connection coefficients are metric-compatible, i.e.,

$$\partial_i g_{jk} - \Gamma_{ij}{}^{l} g_{lk} - \Gamma_{ik}{}^{l} g_{jl} = 0. \tag{57}$$

Writing

$$\lambda = \lambda^{\alpha} G_{\alpha} + \frac{1}{2}\lambda^{\alpha\beta} G_{\alpha\beta} \tag{58}$$

and defining the parameters

$$\xi^j = e^j{}_{\alpha}\lambda^{\alpha}, \quad \epsilon^{\alpha\beta} = \xi^i \Gamma_i{}^{\alpha\beta} - \lambda^{\alpha\beta}, \tag{59}$$

we find that the transformation laws [Eq. (48) and Eq. (49)] are

$$\delta e_i{}^{\alpha} = \xi^j \partial_j e_i{}^{\alpha} + (\partial_i \xi^j) e_j{}^{\alpha} + e_i{}^{\beta}\epsilon_{\beta}{}^{\alpha}, \tag{60}$$

$$\left.\begin{aligned} \delta\Gamma_{i\alpha}{}^{\beta} &= \xi^j \partial_j \Gamma_{i\alpha}{}^{\beta} + (\partial_i \xi^j)\Gamma_{j\alpha}{}^{\beta} - D_i \epsilon_{\alpha}{}^{\beta}, \\ D_i \epsilon_{\alpha}{}^{\beta} &= \partial_i \epsilon_{\alpha}{}^{\beta} - \Gamma_{i\alpha}{}^{\gamma}\epsilon_{\gamma}{}^{\beta} + \epsilon_{\alpha}{}^{\gamma}\Gamma_{i\gamma}{}^{\beta}. \end{aligned}\right\} \tag{61}$$

$$\delta\psi = \xi^j \partial_j \psi + \frac{1}{2}\epsilon^{\alpha\beta} G_{\alpha\beta}\psi. \tag{62}$$

These are identical to the transformation laws for a tetrad, a set of 'spin coefficients' and a field representing the Lorentz group, under the combined action of a Lorentz rotation of the tetrad and a general coordinate transformation.

To justify the assertion that these are indeed the transformation laws appropriate to a gauged Poincaré group, we impose the restrictions

$$e_i{}^{\alpha} = \delta_i{}^{\alpha}, \quad \Gamma_{i\alpha}{}^{\beta} = 0, \tag{63}$$

and consider just those transformations [Eqs. (60)-(61)] that maintain these restrictions. We have $F_{ij} = 0$ and $D_i = \partial_i$, and so

$$\left.\begin{aligned} 0 &= \partial_i \xi_j + \epsilon_{ij}, \\ 0 &= \partial_i \epsilon_{jk}. \end{aligned}\right\} \tag{64}$$

Hence ϵ_{jk} is *constant*, and

$$\xi_j = a_j + \epsilon_{ji} x^i; \tag{65}$$

just the effect of an infinitesimal action of the Poincaré group on *Minkowski* space-time. The field ψ transforms according to

$$\delta\psi = a^j \partial_j \psi + \frac{1}{2}\epsilon^{ij}(x_i \partial_j - x_j \partial_i + G_{ij})\psi, \tag{66}$$

as it should. In terms of the bundle space, we have the left action of the Poincaré group on itself.

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