Quasicrystals and Penrose patterns

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The classification of periodic structures in three dimensions by Fedorov, Schoenflies and Barlow, a century ago, brought the geometrical understanding of perfect crystalline structures into a final, definitive form; there were just 230 possible types, and the topic appeared to be closed. The discovery of quasicrystals in recent years has opened up new and exciting possibilities. Extended structures in space can be orderly and systematic without being periodic. A rather surprising situation surrounding this topic is the amount of theoretical understanding that had already been gained even before any actual quasicrystals were discovered. In particular, Penrose's tiling patterns are two-dimensional quasicrystals. This article introduces the reader to the beautiful geometrical properties of these patterns and describes a three-dimensional generalization.

Quasicrystals

A crystalline structure is an arrangement of atoms that is periodic in three directions—it has three translational symmetries. One can imagine it in terms of a partition of space into identical *unit cells*, each containing an identical arrangement of atoms; the unit cells and their contents are related to one another by translations. In particular, a crystalline structure can be thought of in terms of a *lattice*: the unit cells are then parallelepipeds obtained by partitioning space by three families of equally-spaced planes. For elementary geometrical reasons, the only point symmetry groups that crystalline structures can have are subgroups of the symmetry group of a cube or a hexagonal prism; in particular, they cannot have any *fivefold* symmetry axis.

In 1984, an alloy of aluminium and manganese was discovered[1] whose X-ray diffraction patterns consisted of sharp spots, like the diffraction patterns produced by crystals. But the point symmetry of the atomic arrangement, that these diffraction patterns indicated, was the symmetry of an icosahedron, which has six *fivefold* axes. So the new alloy (Schechtmanite) could not be a crystal in the usual sense. Since its discovery, other 'quasicrystals, have been found. The fascinating question arises: if the atoms in these substances are not arranged periodically, how *are* they arranged?

By an amazing coincidence (an instance of Jung's 'synchronicity'?), while Schechtman and his colleagues were obtaining the baffling diffraction patterns, Levine and Steinhardt[2] had been, quite independently, investigating some intriguing geometrical structures, and computing diffraction patterns from them. They turned out to be practically identical to the diffraction patterns of the new alloy. It is now generally accepted that a quasicrystal can be understood as a systematic (but not periodic) filling of space by unit cells of *more than one kind*. The Steinhardt-Levine structure is a space-filling by two kinds of rhombohedron. It is a three-dimensional generalization of an aperiodic tiling pattern discovered by Penrose.

Periodic tiling patterns

The rich decorative possibilities of *periodic* tiling patterns have been explored and exploited by every culture, since the beginnings of civilization. The ingenuity and inventiveness of medieval Islamic architects is especially remarkable, and often quite astonishing[3–5]. Some of their geometrical methods have recently been rediscovered by Chorbachi[6]. The example shown in Figure 1 occurs in the mausoleum of I'tamad al-Dawla, at Agra. It employs four tile shapes. The dotted line indicate a unit cell.

Aperiodic tiling patterns

An *aperiodic* set of tiles is a set of shapes with the property that, though the whole Euclidean plane can be

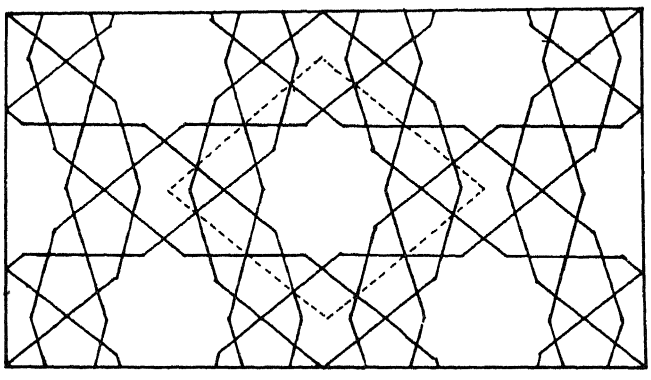

Figure 1.

covered by non-overlapping replicas of the shapes, no periodically repeating tiling pattern can be constructed from them. In 1966, Berger[7] demonstrated the existence of an aperiodic set of tiles, thereby disproving 'Wang's conjecture' that any set of tiles capable of tiling the whole plane could tile it periodically. Berger's first aperiodic set contained over 20,000 tile shapes. He was later able to reduce the number to 104. In 1971, Robinson[8] discovered an aperiodic set consisting of only 6 shapes; an alternative aperiodic set of six tiles was found independently by Penrose[9] in 1974. Finally, Penrose found aperiodic sets with only *two* different shapes[10,11].

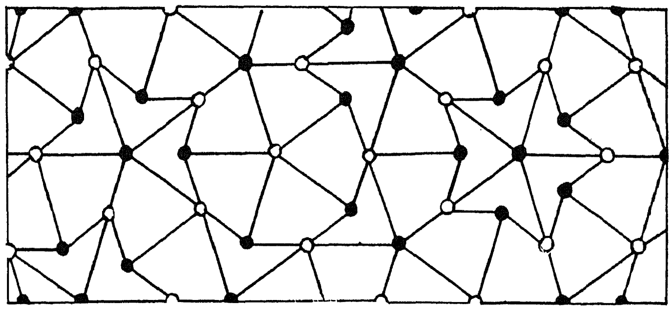

Figure 3.

Kites and darts

Two quadrilateral tile shapes (a *kite* and a *dart*) can be obtained by dissecting a regular decagon as in Figure 2. The subdivisions of the decagon radii are 'golden sections'. The golden ratio $1:\tau$ is defined by the golden number $\tau=\frac{1}{2}(1+\sqrt{5})=2\cos(\pi/5)=1.6180\ldots$, which satisfies the important relations $\tau^2=\tau+1$ and $\tau^{-1}=\tau-1$. The golden ratio is the ratio of the side of a regular decagon to its circumradius, and also the ratio of the side of a regular pentagon to its diagonal.

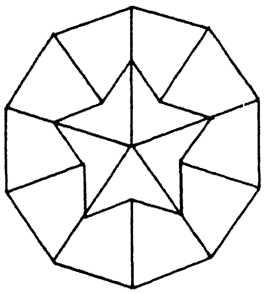

Figure 2.

Of course, many periodic tiling patterns can be constructed from the kites and darts. Penrose's *aperiodic* kite and dart patterns are formed by marking the tiles in some way and introducing a *matching rule*, to be obeyed when composing a pattern, that forbids periodic arrangements. The simplest way of doing this is to distinguish two kinds of vertex (red and green vertices, or more conveniently, black and white vertices) and to insist that when patterns are built up from the tiles, vertices have to match when placing two tiles edge to edge (Figure 3). A pattern constructed in compliance with this matching rule is a 'Penrose kite and dart pattern'. An alternative, but equivalent, way of forbidding periodicity is to abandon straight lines for the tile edges so that the tiles interlock like jigsaw puzzle pieces. Using this approach, Penrose was able, with considerable ingenuity, to modify the kite and dart patterns to obtain aperiodic patterns of interlocking tiles in the shape of two chickens! (Figure 4).

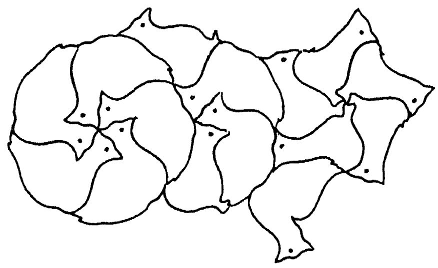

Figure 4.

Another way of marking the kites and darts is shown in Figure 5 (the subdivisions used in constructing the markings are golden sections). When a Penrose kite and dart pattern is marked in this way, the markings form the edges of another Penrose kite and dart pattern, with smaller tiles (the reduction factor is $\tau^{-1}=\frac{1}{2}(\sqrt{5}-1)$); we have a *decomposition rule* for Penrose kite and dart patterns. *Inflation* of a pattern is the simultaneous application of decomposition and expansion by a factor τ. By repeated inflation of any patch of a pattern, one obtains a sequence of larger and larger patches, the size of the tiles remaining the same. This establishes that the whole of the plane can be covered by a Penrose kite and dart pattern. The inverse of decomposition is *recomposition*. The markings on the tiles in this case are the short diagonals of the darts and the long edges of the kites; when a Penrose kite and dart pattern is marked in this way the markings form the edges of

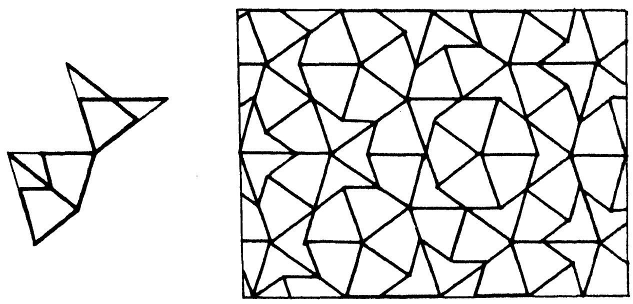

Figure 5.

another such pattern, with larger tiles (the enlargement factor is τ). This enables us to prove that every Penrose kite and dart pattern is aperiodic: Suppose there existed a periodic Penrose kite and dart pattern with a period length l. Recomposition would give another periodic pattern with a period length l, but with bigger tiles; continued iteration would eventually give a periodic Penrose kite and dart pattern, with period length l but with tile edges longer than l—which is absurd! Hence the kite and dart, with an appropriate matching rule, is an aperiodic set. In particular, Penrose's two chickens are an aperiodic set of tile shapes.

The Penrose kite and dart patterns have been studied extensively. The reader who wishes to find out more about their strange properties should consult Martin Gardner's lucid *Scientific American* article[11], or the chapter on aperiodic tilings in Grünbaum and Shephard's book *Tilings and Patterns*[12].

Penrose rhombs

Let the kites and darts of a Penrose pattern be marked as in Figure 6. The markings then form the edges of a different aperiodic tiling pattern, consisting of two kinds of rhombus: a 'fat rhomb' with an angle $2\pi/5$ and a 'thin rhomb' with an angle $\pi/5$. A matching rule to be obeyed when building a pattern from these two rhombs can be imposed by putting single arrowheads and double arrowheads on the tile edges as in Figure 7, and insisting that arrowheads have to match when tiles are placed edge to edge.

We have shown how a rhomb pattern can be derived from a kite and dart pattern. The inverse procedure is simple: the long diagonals of the fat rhombs and the double-arrowheaded edges of the thin rhombs form the edges of a kite and dart pattern. This correspondence between the two kinds of pattern shows that the two (marked) rhombs constitute an aperiodic set.

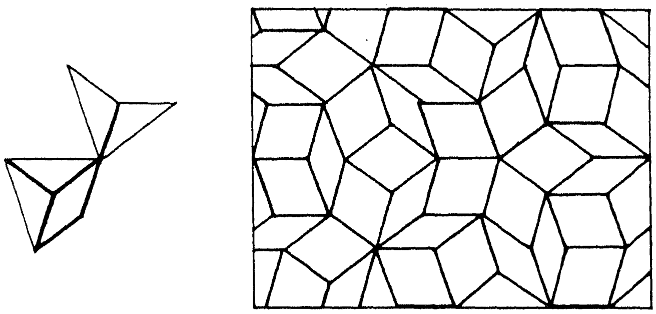

Figure 6.

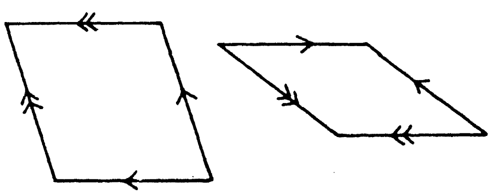

Figure 7.

When building up a Penrose pattern, a degree of arbitrary choice is involved in the placing of tiles: *randomness* is a characteristic feature of the structure of these aperiodic patterns. But the structure is also highly systematic. It is the complex interplay between randomness and order that makes these patterns so intriguing.

Observe that the tile edges of a Penrose rhomb pattern have only five possible orientations. The five families of parallel edges are parallel to the five symmetry axes of a regular pentagon. The patterns are said to have *long-range orientational order*, with fivefold symmetry. Note that fivefold symmetry is impossible in a *periodic* two-dimensional pattern. We have here a clue to the understanding of the nature of the fivefold axes of quasicrystals.

If all the edges carrying double arrowheads are deleted from a rhomb pattern, what remains is an aperiodic tiling pattern formed from a set of *three* different tiles (Figure 8), which we shall call starfish, ivy leaf and hex. The matching rule for assembling patterns from this set is provided by the arrowheads on their edges (Figure 9). Restoration of the rhomb pattern by re-inserting the lines with double arrowheads is obvious. It follows that any Penrose rhomb can be obtained by assembling prefabricated 'supertiles': a starfish consisting of five fat rhombs, an ivy leaf consisting of three fat rhombs and a thin rhomb, and a hex consisting of a fat rhomb and two thin rhombs.

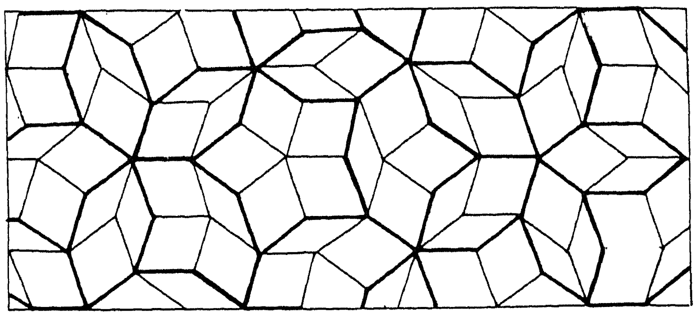

Figure 8.

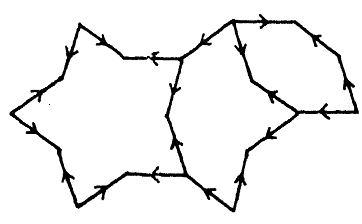

Figure 9.

Penrose's first pattern

The six marked tiles in Figure 10 are an aperiodic set, if we impose the matching rule that the lines marking the

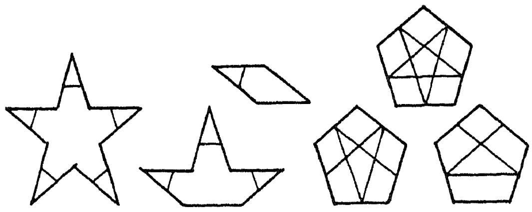

Figure 10.

tiles have to continue across tile boundaries (to form lines that run right through the pattern). This set is equivalent to the first aperiodic set discovered by Penrose, who showed that an equivalent matching rule can be imposed by giving the tiles wiggly interlocking edges, instead of marking them[9].

Figure 11 shows a patch of pattern formed from the marked set, with markings deleted for clarity. Figure 12 reproduces a diagram from Kepler's book *Harmonice Mundi*, published in 1619. The resemblance is surprising. Even more surprising are Kepler's remarks about his tiling scheme: he concluded that the pattern would never repeat, there would always be "surprises"[13]. The four shapes in Kepler's pattern are not in fact an aperiodic set; a periodic pattern using the same shapes can be devised[12]. Nevertheless, it cannot be denied that Kepler had anticipated the concept of aperiodic tiling patterns, by about 350 years!

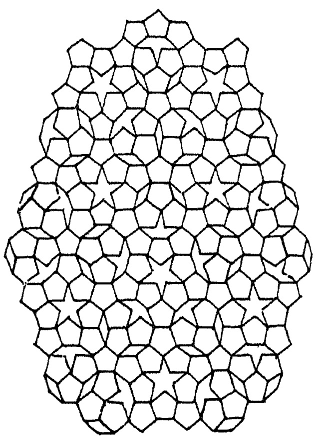

Figure 11.

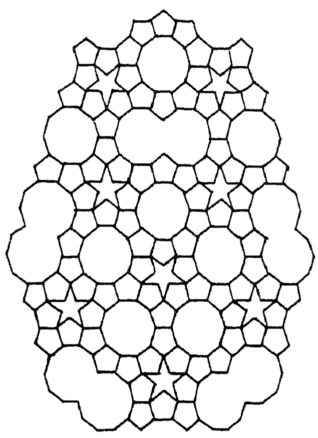

Figure 12.

Penrose's six marked tiles lead to aperiodic patterns consisting of four tile shapes: a pentagon, a five-pointed star, a rhombus, and "a kind of three-pointed half-star"[9]. These aperiodic patterns can alternatively be derived from the rhomb patterns by marking all the rhombs as in Figure 13. The construction of the markings uses golden section of two of the edges of each rhomb and perpendicular bisection of the other two edges. Conversely, a rhomb pattern can be recovered from a pattern of the kind we have introduced: join the mid-points of every pair of adjacent pentagons; we get a pattern of starfish, ivy leaves and hexes, which can then be dissected into rhombs.

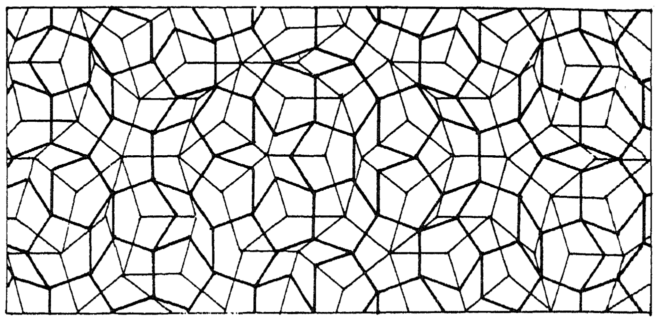

Figure 13.

A one-dimensional quasicrystal

Take a two-dimensional lattice of squares, and cut from it a strip exactly wide enough to contain one of the squares (Figure 14). Project all the lattice points contained in the strip on to a line parallel to the strip. The result is a sequence of intervals on the line, of two different lengths. It is quite clear that if the tangent of the angle between the strip and a lattice direction is irrational, *the sequence will never repeat*, however long we make the strip. We have an aperiodic tiling of a one-dimensional space, using two kinds of tile. A particularly important and interesting case arises when the angle of the strip is chosen so that the ratio of lengths of the two 'tiles' is the golden number τ. Denoting the length of the long tile by A, and the length of the short tile by B, we get an aperiodic sequence... $ABAAB$ $ABAABAAB$.... It is called the *Fibonacci sequence*. Consider the *distance* of a particular lattice point from the top edge of the strip. It is clear that no other lattice point can be at *precisely* that distance—that would imply that the sequence repeats. But if we simply ask for another lattice point whose distance approximates to that of the given point, within an arbitrarily chosen degree of accuracy, there will always be an *infinite number* of such points. It follows from this that the Fibonacci sequence has a very strange property: any portion of the sequence will occur again somewhere in the sequence, at another position (in fact, at an infinite number of other positions!). This result has an analogue for two-dimensional Penrose patterns; if we take any finite patch of an infinite Penrose pattern, exact replicas

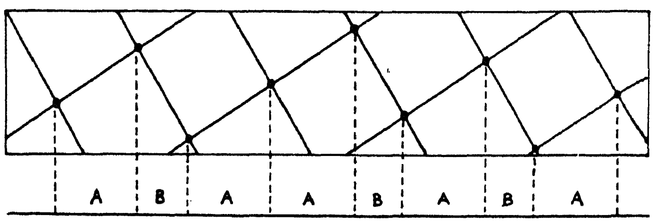

Figure 14.

of it must occur (infinitely many times) somewhere else in the pattern!

Note that moving the strip in Figure 14 relative to the lattice, keeping it parallel to its original position, simply *shifts* the sequence. The pattern of *A*s and *B*s on the line is replaced by a different part of the same infinitely long Fibonacci sequence.

The Fibonacci sequence can be generated iteratively by starting from the single symbol A and then applying the rule $A \rightarrow AB$, $B \rightarrow A$. We get

$$A,\ AB,\ ABA,\ ABAAB,\ ABAABABA, \ldots,.$$

This is a *decomposition rule* for the one-dimensional tiling patterns, analogous to the decomposition rules for Penrose tilings of the plane.

The sequence is called the Fibonacci sequence for the following reason. The term Fibonacci sequence is also used to denote the number sequence 1 1 2 3 5 8 13 21... in which each term is the sum of the two preceding terms. (The ratio of two successive terms approaches τ, in the limit). In 1202, Fibonacci discussed this number sequence in connection with the proliferation of rabbits. Let A denote a pair of adult rabbits, and let B denote two baby rabbits. Suppose that, in some fixed period of time, every adult pair gives birth to two babies. ($A \rightarrow AB$) and every baby becomes adult ($B \rightarrow A$). This rule is identical to the decomposition rule for generating the sequence of one-dimensional tiles! The number sequence gives the total number of pairs, after the elapse of 1, 2, 3,... time periods (starting from two baby rabbits).

When the tiles in figure 10 are assembled into a pattern, the marks become lines running across the pattern. These lines are called *Amman bars.* We get five families of parallel Amman bars, exhibiting the long-range orientational order of the pattern. The bars of one family are not equally spaced; there is a wide spacing A and a narrow spacing B that occur in a seemingly random sequence (Figure 15). But the sequence is not in a fact random; it is a *Fibonacci sequence.* One can relate the Amman bars to the Penrose rhomb patterns. We then find that all the fat rhombs of the pattern are marked identically by the bars, as are all the thin rhombs. The rhombs with

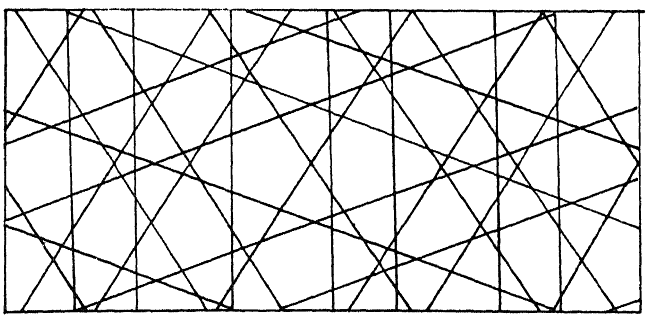

Figure 15.

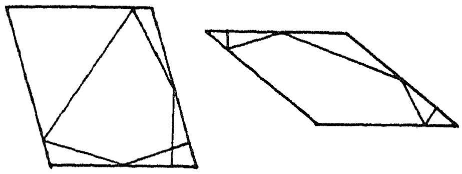

Figure 16.

Amman bar markings are shown in Figure 16.

The fact that any Penrose tiling pattern has this hidden structure revealed by the Fibonacci sequences of its associated Amman bars, is acknowledged in the terminology: the Penrose patterns are now usually referred to as 'quasiperiodic' rather than simply 'aperiodic'.

The various analogies that exist between the Fibonacci sequence and Penrose tiling patterns are not fortuitous. They are related to the fact that the Penrose rhomb patterns can be obtained as projections from a slice of a hypercubic lattice in five dimensions!

Patterns in three dimensions

An icosahedron has 6 fivefold axes, 10 threefold axes, 15 twofold axes and 15 reflection planes.

The 30 midpoints of the edges of an icosahedron are the vertices of an icosidodecahedron; its faces are twenty equilateral triangles and twelve regular pentagons. The *rhombic triacontahedron* is its dual (Figure 17).

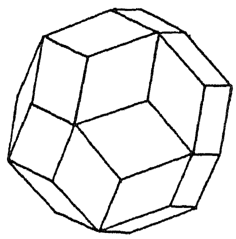

Figure 17.

Through every edge of the icosahedron, imagine a plane perpendicular to the radius through the midpoint of the edge; the thirty planes cut out a rhombic triacontahedron. Its thirty faces are congruent rhombuses with diagonals in the ratio $1:\tau$. Fifteen pairs of opposite faces are parallel to the fifteen reflection planes; every edge is parallel to one of the fivefold axes; the faces having an edge parallel to a given fivefold axis form a zone of ten faces. The face diagonals are parallel to the twofold axes.

A *rhombohedron* is a parallelepiped bounded by six congruent rhombuses. It has two opposite vertices at which the three face angles are equal (i.e. it has a threefold symmetry axis). It is said to be acute or obtuse according to the nature of these angles[14].

A rhombic triacontahedron can be built out of

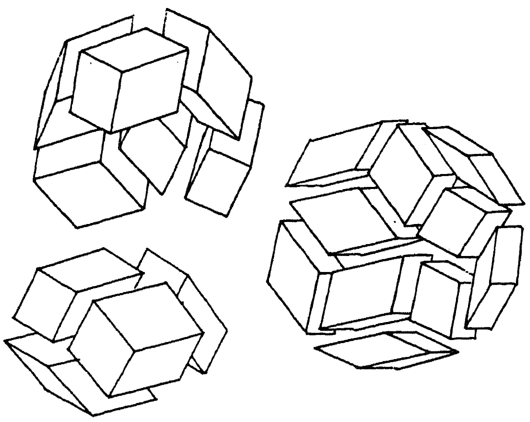

Figure 18.

twenty rhombohedra, of two kinds; ten acute and ten obtuse (Figure 18). This was pointed out by Kowalewski in 1935[14]. The two rhombohedral units will be called Kowalewski units. They can be constructed as follows: Choose any three of the fivefold axes of an icosahedron. For a given edge length, there is a unique rhombohedron with edges parallel to these three axes. This is one of the Kowalewski units. The other is obtained from the other three fivefold axes (Figure 19). Curiously, it does not matter how the set of six axes is split into two sets of three: we always get the same two shapes differently oriented. The faces of the two Kowalewski units are rhombuses with diagonals in the ratio $1:\tau$.

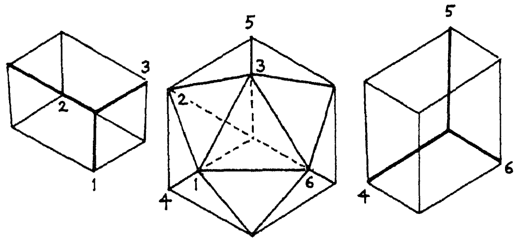

Figure 19.

According to MacKay[15], it was Robert Amman who first recognized that the two Kowalewski units are the three-dimensional analogues of the Penrose rhombs. From what we have said above, it should be apparent that, if we build a three-dimensional tiling pattern from Kowalewski units placed face to face, all the faces will be parallel to reflection planes of an icosahedron, all the edges will be parallel to fivefold axes, all the face diagonals will be parallel to twofold axes and all the symmetry axes of the units will be parallel to threefold axes. Nets for an *aperiodic set* of four marked units[16], two acute and two obtuse, are shown in Figure 20 (white circles have to coincide with black circles when two units are placed face to face). The Levine-Steinhardt models are aperiodic tilings of space by Kowalewski units. They have a *long range orientational order with icosahedral symmetry*. They also have a *quasiperiodic* property, characterized by parallel sets of 'Amman planes' with Fibonacci spacing.

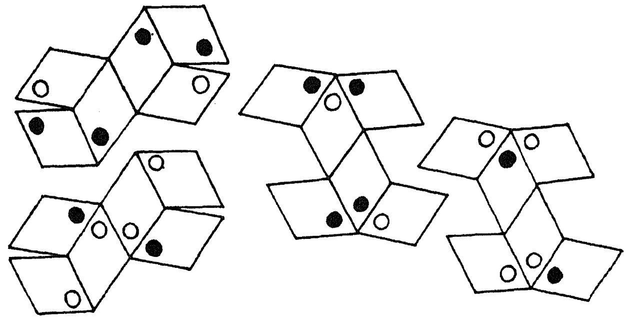

Figure 20.

A quasiperiodic structure can account for the sharp spots in the diffraction patterns of quasicrystals. Recall that the sharp spots in diffraction patterns of ordinary crystals are a manifestation of the periodic structure; they arise from constructive interference of X-rays reflected from equally-spaced parallel layers of atoms, at certain angles (Bragg diffraction). Fourier analysis shows that the same thing happens for layers spaced quasiperiodically—in particular, for layers with Fibonacci spacing.

A crystal in six dimensions

Fivefold symmetry is forbidden for periodic structures in two and three dimensions. This is not so in higher dimensions. For example, the set of all points whose coordinates are integers is a cubic lattice. In five dimensions, a fivefold symmetry of the cubic (more correctly, hypercubic) lattice is given by cyclically permuting the five coordinates. The symmetry transformations of an icosahedron permute its six fivefold axes. We can define a group of rotations and reflections in a six-dimensional space by simply applying the same permutations to the six coordinates. Thus, the cubic lattice in six dimensions has an icosahedral symmetry. This idea leads to an important method of computing the vertices of a Levine-Steinhardt pattern. We have seen how a one-dimensional quasicrystal can be obtained by projecting a strip of a two-dimensional square lattice on to a one-dimensional space. Similarly, a three-dimensional quasicrystal (a Levine-Steinhardt pattern of Kowalewski units) can be obtained by projecting a slice of a six-dimensional lattice on to a three-dimensional space[17].

The twelve points in 3-space with coordinates $(0\ \pm 1\ \pm\tau)$, $(\pm\tau\ 0\ \pm 1)$, $(\pm 1\ \pm\tau\ 0)$ are the vertices of an icosahedron. Pick out six of them to represent the fivefold axes and write them as columns of a matrix, for example

$$A = v\begin{bmatrix} \tau & 0 & -\tau & -1 & 1 & 0 \\ 0 & -1 & 0 & \tau & \tau & 1 \\ 1 & \tau & 1 & 0 & 0 & \tau \end{bmatrix}.$$

Now think of the *rows* of this matrix as vectors in a six-dimensional space. If the normalization factor is chosen to be $y = 1/[2(1+\tau^2)]^{\frac{1}{2}}$, they are orthonormal. They lie in a three-dimensional space which we call *p*-space. The matrix

$$P = A^T A = \frac{1}{2\sqrt{5}}\begin{bmatrix} \sqrt{5} & 1 & -1 & -1 & 1 & 1 \\ 1 & \sqrt{5} & 1 & -1 & -1 & 1 \\ -1 & 1 & \sqrt{5} & 1 & -1 & 1 \\ -1 & -1 & 1 & \sqrt{5} & 1 & 1 \\ 1 & -1 & -1 & 1 & \sqrt{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & \sqrt{5} \end{bmatrix}.$$

projects points of the six-dimensional space on to *p*-space. The images of the six coordinate axes are six lines arranged like the fivefold axes of an icosahedron. The image of a unit cube in 6-space, parallel to the cubes of the lattice of points with integer coordinates, is a *rhombic triacontahedron* in *p*-space! If we eliminate from the six-dimensional lattice all the vertices that project to points outside this triacontahedron, we are left with a *slice* of the six-dimensional lattice. It is analogous to the strip of two-dimensional lattice in Figure 14. The three-dimensional space orthogonal to *p*-space will be called *q*-space. Projection on to *q*-space is achieved by the matrix $Q = I - P$. The image in *q*-space of any three-dimensional facet (ordinary cube) of a lattice hypercube is a Kowalewski unit in *q*-space. Moreover, it turns out that the projection on to *q*-space of all the three-dimensional facets contained in the slice gives a Levine–Steinhardt aperiodic pattern in *q*-space!

A similar method exists for computing Penrose rhomb patterns, by projection on to a plane of a five-dimensional cubic lattice. We simply start with a 2 × 5 matrix *A* whose columns are the coordinates of the vertices of a regular pentagon centred at the origin.

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Optimum nutrition

Mahtab S. Bamji

Recommended dietary allowances, or dietary standards, are amounts of essential nutrients considered necessary for normal physiology and health of individuals of a defined population. But the question of optimum nutrition is complex, and the current recommendations are at best tentative.

Nutrition has been defined as the 'science of food as it relates to optimal health and performance'[1]. Though mankind has always been interested in food which would ensure health, reproductive vigour and well-being, both of human beings and domestic animals, it was only towards the turn of this century that the subject of nutrition shifted from the realm of mere beliefs and dogmas to a systematic science. Discovery of vitamins and minerals as essential food factors, enzymes as biocatalysts, and the elucidation of metabolic pathways laid the foundation of both the sciences of biochemistry as well as nutrition. Till the middle of the present century, the two sciences were

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