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Geometry of the Mathieu groups and Golay codes

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Abstract. A brief review is given of the linear fractional subgroups of the Mathieu groups. The main part of the paper then deals with the projective interpretation of the Golay codes; these codes are shown to describe Coxeter's configuration in $PG(5,3)$ and Todd's configuration in $PG(11,2)$ when interpreted projectively. We obtain two twelve-dimensional representations of M_{24}. One is obtained as the collineation group that permutes the twelve special points in $PG(11,2)$; the other arises by interpreting geometrically the automorphism group of the binary Golay code. Both representations are reducible to eleven-dimensional representations of M_{24}.

Keywords. Geometry; Mathieu groups; Golay codes; Coxeter's configuration; hemi-icosahedron; octastigms; dodecastigms.

1. Introduction

We shall first review some of the known properties of the Mathieu groups and their linear fractional subgroups. This is mainly a brief resumé of the first two of Conway's 'Three Lectures on Exceptional Groups' [3] and will serve to establish the notation and underlying concepts of the subsequent sections.

The six points of the projective line $PL(5)$ can be labelled by the marks of $GF(5)$ together with a sixth symbol ∞ defined to be the inverse of 0. The symbol set is $\Omega = \{0\,1\,2\,3\,4\,\infty\}$. The group $L_2(5)$ of homographies on the line is generated by the modulo 5 transformations on Ω:

$$\alpha{:}i \to i+1, \quad \beta{:}i \to -i, \quad \gamma{:}i \to -1/i. \tag{1.1}$$

As permutations,

$$\begin{aligned} \alpha &= (0\,1\,2\,3\,4) \\ \beta &= (14)(23) \\ \gamma &= (0\,\infty)(14). \end{aligned} \tag{1.2}$$

The group S_6 of all permutations of the six symbols is generated by α and τ, where

$$\tau = (0\,\infty). \tag{1.3}$$

(As a modulo 5 transformation, $\tau{:}\, i \to 1/i^3$). Alternatively, the S_6 is generated by α and π, where

$$\pi = (0\,\infty)(14)(23) \tag{1.4}$$

($\pi : i \to -1/i^3$). The mapping θ on S_6 defined by

$$\alpha^\theta = \alpha \quad \tau^\theta = \pi \tag{1.5}$$

is an involutory outer automorphism of S_6. It is obviously not an inner automorphism because the fifteen transpositions correspond, under θ, to the fifteen regular involutions (products of three mutually commuting transpositions). These correspondences are the same as the correspondences between Sylvester's *duads* and *synthemes* of six symbols [1], [2], [6]. Any transposition appears in the structure of three of the regular involutions and each regular involution is comprised of three transpositions—a Cremona–Richmond 15_3. The self-dual property of the 15_3 corresponds to the involutory property of θ ($\theta^2 = 1$).

The S_6 can alternatively be generated by the five transpositions $(0\,\infty)$, $(1\,\infty)$ $(2\,\infty)$ $(3\,\infty)$, $(4\,\infty)$ or by the five regular involutions

$$\begin{aligned}
\pi_0 &= (\infty\,0)(14)(23),\\
\pi_1 &= (\infty\,1)(20)(34),\\
\pi_2 &= (\infty\,2)(31)(40),\\
\pi_3 &= (\infty\,3)(42)(01),\\
\pi_4 &= (\infty\,4)(03)(12),
\end{aligned} \tag{1.6}$$

(obtained by acting on π with the powers of α).

Under the action of θ, the generators α, β, γ of $L_2(5)$ are mapped to α, β, δ, where

$$\delta = (12)(34). \tag{1.7}$$

Whereas the original $L_2(5)$, generated by α, β, γ is transitive on the six symbols, the $L_2(5)$ generated by α, β, δ acts on only five of them. It is in fact the alternating group A_5 of even permutations of $\{0\,1\,2\,3\,4\}$. We have the isomorphism $L_2(5) \sim A_5$. The action of α, β, γ on the set Ω induces the action of α, β, δ on the set of five objects (1.6).

$$(\pi_i)^\alpha = \pi_{i\alpha}, \quad (\pi_i)^\beta = \pi_{i\beta}, \quad (\pi_i)^\gamma = \pi_{i\delta} \tag{1.8}$$

(the notation here is $\pi^\alpha = \alpha^{-1}\pi\alpha$, etc).

The four permutations α, β, γ and δ together generate the group A_6 of all even permutations of six symbols. Indeed, this A_6 is generated by α, β, and δ only, since $\beta = (\gamma\delta)^2$.

The S_6 together with its outer automorphism θ generates a group $S_6 2$ (a group with an invariant subgroup S_6 of index 2).

The eight points of the projective line $PL(7)$ can be labelled by the set $\Omega = \{0\,1\,2\,3\,4\,5\,6\,\infty\}$ consisting of the marks of $GF(7)$ and the extra symbol ∞. The group $L_2(7)$ of homographies on the line is generated by the modulo 7 transformations

$$\alpha : i \to i+1, \quad \beta : i \to 2i, \quad \gamma : i \to -1/i. \tag{1.9}$$

As permutations,

$$\alpha=(0\,1\,2\,3\,4\,5\,6),$$
$$\beta=(124)(365),$$
$$\gamma=(0\,\infty)(16)(23)(45). \tag{1.10}$$

Analogously to the case of $PL(5)$, there exists another $L_2(7)$ that acts on only seven of the eight symbols; it is generated by α, β and

$$\delta=(12)(36). \tag{1.11}$$

The two groups $L_2(7)$ are mapped to each other by an involutory outer automorphism of the group F generated by α, β, γ and δ. In Figure 1, the vertices of Fano's configuration 7_3 have been labelled by the marks of $GF(7)$ (observe that α cyclically permutes the vertices of the 'self-inscribed and circumscribed heptagon' 0 1 2 3 4 5 6, and that β rotates the figure and δ reflects it). The projective plane $PG(2,2)$ is a Fano 7_3 and α, β and γ generate its group of collineations—the simple group $L_3(2)$ of order 168. We have the isomorphism $L_2(7)\sim L_3(2)$.

The quadratic residues of $GF(7)$ are the elements of the set $Q=\{0\,1\,2\,4\}$. The complementary set is $N=\{3\,5\,6\,\infty\}$. Consider the seven pairs of four symbols obtained by operating with the powers of α on Q, N:

$$\begin{aligned}
P_0&=\{0\,1\,2\,4\}\{3\,5\,6\,\infty\},\\
P_1&=\{1\,2\,3\,5\}\{4\,6\,0\,\infty\},\\
P_2&=\{2\,3\,4\,6\}\{5\,0\,1\,\infty\},\\
P_3&=\{3\,4\,5\,0\}\{6\,1\,2\,\infty\},\\
P_4&=\{4\,5\,6\,1\}\{0\,2\,3\,\infty\},\\
P_5&=\{5\,6\,0\,2\}\{1\,3\,4\,\infty\},\\
P_6&=\{6\,0\,1\,3\}\{2\,4\,5\,\infty\}.
\end{aligned} \tag{1.12}$$

The action of the $L_2(7)$ generated by α, β, γ, on Ω, induces the action of the $L_2(7)$ generated α, β, δ on this set of seven objects;

$$\alpha{:}P_i\to P_{i\alpha},\quad \beta{:}P_i\to P_{i\beta},\quad \gamma{:}P_i\to P_{i\delta}. \tag{1.13}$$

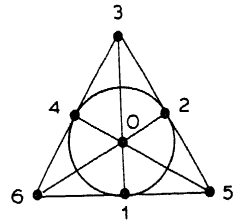

Figure 1.

The seven objects P_i are the seven pairs of complementary tetrads of a Steiner system $S(3, 4, 8)$. (Conway employs the involution $\pi = (\infty 0)(13)(26)(45)$ instead of our object Q, N. The effect is the same. Our seven objects P_i are more suitable for our later purposes).

The twelve points of $PL(11)$ can be labelled by the symbol set $\Omega = \{0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,X\,\infty\}$. The first eleven of these symbols are the marks of $GF(11)$. The homography group $L_2(11)$ is generated by

$$\alpha: i \to i+1, \quad \beta: i \to 3i, \quad \gamma: i \to -1/i \tag{1.14}$$

(modulo 11), i.e. by the permutations

$$\begin{aligned} \alpha &= (0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,X), \\ \beta &= (1\,3\,9\,5\,4)(8\,2\,6\,7\,X), \\ \gamma &= (0\,\infty)(1X)(25)(37)(48)(69). \end{aligned} \tag{1.15}$$

The set $Q = \{0\,1\,3\,9\,5\,4\}$ is the set of quadratic residues. The non-zero quadratic residues are the powers of 3 modulo 11. The complementary set is $N = \{8\,2\,6\,7\,X\,\infty\}$. Observe that β leaves invariant these two hexads and γ interchanges them.

The Mathieu group M_{12} acting on the twelve symbols is generated by $\alpha, \beta, \gamma, \Delta$, where

$$\Delta = (2X)(34)(59)(67). \tag{1.16}$$

As is well known, M_{12} is the automorphism group of the Steiner system $S(5, 6, 12)$. The 66 complementary pairs of hexads are the images under the action of M_{12} of the generic hexad pair Q, N. Observe that $\beta = (\gamma\Delta)^6$ so in fact M_{12} is generated by α, γ, Δ. Our original subgroup $L_2(11)$ was generated by α, β, γ. Another $L_2(11)$ subgroup is generated by α, β, Δ. The original $L_2(11)$ is transitive on the 12 symbols while the other $L_2(11)$ acts on only 11 of them. This is analogous to the previous cases. However, the $L_2(11)$ generated by α, β, γ is maximal in M_{12}, while the $L_2(11)$ generated by α, β, Δ is a subgroup of M_{11}, so there can be no question of an automorphism of M_{12} mapping these two groups onto each another. On the other hand we have an automorphism θ defined by

$$\alpha^\theta = \alpha^{-1}, \quad \beta^\theta = \beta, \quad \gamma^\theta = \gamma, \quad \Delta^\theta = \Delta. \tag{1.17}$$

The 24 points of the projective line $PL(23)$ can be labelled by $\Omega = \{0\,1\,2\,3 \ldots 22\,\infty\}$ consisting of the marks of $GF(23)$ and the symbol ∞. The homography group $L_2(23)$ is generated by

$$\alpha: i \to i+1, \quad \beta: i \to 2i, \quad \gamma: i \to -1/i \tag{1.18}$$

(modulo 23). As permutations,

$$\begin{aligned} \alpha &= (0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14\,15\,16\,17\,18\,19\,20\,21\,22), \\ \beta &= (1\,2\,4\,8\,16\,9\,18\,13\,3\,6\,12)(5\,10\,20\,17\,11\,22\,21\,19\,15\,7\,14), \\ \gamma &= (0\,\infty)(1\,22)(2\,11)(4\,17)(8\,20)(16\,10)(9\,5)(18\,14)(13\,7)(3\,15)(6\,19)(12\,21) \end{aligned} \tag{1.19}$$

The group is extended to the Mathieu group M_{24} by the supplementary generator

$$\delta = (7\;5\;21\;20\;10)(22\;14\;17\;11\;19)(13\;9\;12\;8\;16)(1\;18\;4\;2\;6). \qquad (1.20)$$

In fact M_{24} is generated by α, β, and $\gamma\delta$, since $(\gamma\delta)^5 = \gamma$ and $(\gamma\delta)^6 = \delta$.
We have

$$\gamma\delta = (\infty\;0)(3\;15)(1\;14\;4\;11\;6\;22\;18\;17\;2\;19)(5\;12\;20\;16\;7\;9\;21\;8\;10\;13). \qquad (1.21)$$

The generators α, β, and δ generate M_{23} (the one-point stabilizer fixing the symbol ∞). M_{24} is the automorphism group of the Steiner system $S(5, 8, 24)$. The automorphism group is transitive on the 759 special octads of the Steiner system, and on the 2576 umbral dodecads (an umbral dodecad is constructed as the symmetric difference of two special octads with just two common symbols). Denoting the quadratic residue set by $Q = \{0\;1\;2\;4\;8\;16\;9\;18\;13\;3\;6\;12\}$ and the complementary set by N, then Q, N is a generic pair of complementary umbral dodecads of the $S(5, 8, 24)$. A complementary pair of umbral dodecads is called a 'duum'.

2. Correspondences

We have mentioned that S_6 is extended to a group $S_6 2$ by including the automorphism θ of S_6 as a supplementary generator. This group $S_6 2$ can be realized as a permutation group on twelve symbols. In fact, $S_6 2$ occurs in M_{12} as the stabilizer of a complementary hexad pair in $S(5, 6, 12)$. To see this, set up correspondences between $\Omega(5)$ and $Q(11)$, and between $\Omega(5)$ and $N(11)$, as follows

Ω:	0	1	2	3	4	∞
$Q(11)$:	1	3	9	5	4	0
$N(11)$:	8	2	6	7	X	∞

(2.1)

(i.e. $i \to 1/4^i$ and $i \to -3^{i+1}$ modulo 11). The permutations of S_6 on $\Omega(5)$ now induce permutations on $Q(11)$ and on $N(11)$. Consider the effect of acting on $Q(11)$ and $N(11)$ simultaneously with the two inequivalent representations of S_6. Thus the α of S_6 is realised on $\Omega(11)$ as the permutation

$$(1\;3\;9\;5\;4)(8\;2\;6\;7\;X), \qquad (2.2)$$

which is just the β of M_{12}. The permutation τ of S_6 is realised on $\Omega(11)$ by the induced action of τ on $Q(11)$ and π on $N(11)$:

$$(8\;\infty)(1\,0)(6\,7)(2\,X). \qquad (2.3)$$

This permutation also belongs to M_{12} (by a time-consuming process of trial and error and intuition, it was found to be Δ^η, where $\eta = \alpha^{-2}\gamma\alpha^3\beta^2$. There may be a simpler expression. A straightforward test for whether any particular permutation on $\Omega(11)$ belongs to M_{12} will be revealed in § 4). Finally, the automorphism θ of S_6 is realised on

$\Omega(11)$ by an interchange of $Q(11)$ and $N(11)$:

$$(18)(32)(96)(57)(4X)(0\,\infty). \tag{2.4}$$

This also belongs to M_{12}. As is easily verified, it is $(\gamma\Delta)^5$.

Incidentally: under the above correspondence, the β, γ and δ of S_6 induce, respectively, $\Delta, \Delta^{\alpha^{-3}}$ and $\Delta^{\alpha^{-3}(\alpha\Delta)^5}$ of M_{12}.

Analogously to the above, M_{12} can be extended to $M_{12}2$ by incorporating the automorphism θ of M_{12} as a supplementary generator, and this $M_{12}2$ occurs as a subgroup of M_{24} in the role of stabilizer of a duum of the Steiner system $S(5, 8, 24)$. To see how this works, set up the following correspondences $\Omega(11) \to Q(23)$ and $\Omega(11) \to N(23)$:

$\Omega(11)$:	0	1	2	3	4	5	6	7	8	9	X	∞	
$Q(23)$:	3	13	18	9	16	8	4	2	1	12	6	0	(2.5)
$N(23)$:	15	7	14	5	10	20	17	11	22	21	19	∞	

($i \to -2^{i+3}$ and $i \to 1/2^{i+3}$). Realize M_{12} as permutations on $\Omega(23)$ by inducing the actions of $\alpha^{-1}, \beta, \gamma, \Delta$ of M_{12} on $N(23)$, accompanied by the actions of $\alpha, \beta, \gamma, \Delta$ on $Q(23)$. Then the α and β of M_{12} induce, respectively, the permutations β and δ of M_{24}. The permutations induced by γ and Δ of M_{12} also belong to M_{24} (a simple test for this will be revealed later). The automorphism θ of M_{12} is realized as an interchange of $N(23)$ and $Q(23)$. It is in fact just the γ of M_{24}.

A set of three special octads of $S(5, 8, 24)$, with no symbol in common, is called a trio. $L_2(7)$ occurs as a subgroup of M_{12}, as a subgroup of the stabilizer of a trio. As a generic trio, we can take the three octads chosen by Curtis for the construction of the miracle octad generator (MOG) [4][5]. We set up correspondences between $\Omega(7)$ and each of these three octads, as follows:

$\Omega(7)$:	0	1	2	3	4	5	6	∞	
	0	8	20	14	15	3	18	∞	(2.6)
	4	13	7	11	10	16	2	17	
	9	1	12	22	6	5	21	19	

Then the permutations α, β and γ of $L_2(7)$ induce permutations of $\Omega(23)$, which, as we shall show in § 6, belong to M_{24}. (Quite different $L_2(7)$ subgroups of M_{24} are the octern groups [3][5], which we shall not be concerned with in the present work).

3. The hemi-icosahedron and Coxeter's eleven-celled polytope

The group A_5 is the rotation group of an icosahedron. The representation of it as a transitive permutation group on *six* objects corresponds to the even permutations of the six diameters, brought about by the rotations. Its representation as the group of even permutations of five objects corresponds to the effect of the rotations on either a set of five inscribed cubes, or a set of five inscribed tetrahedra [8], or a set of five circumscribed octahedra [7], or a set of five trios of mutually perpendicular golden rectangles [7] (figure 3).

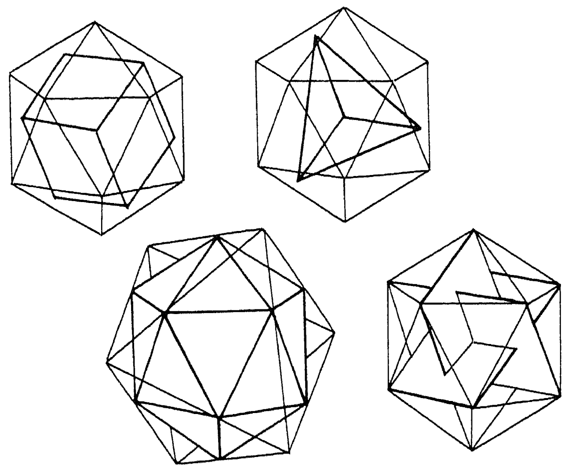

Figure 2.

The full symmetry group of the icosahedron, including reflections, can be generated by three reflections R_1, R_2, R_3 satisfying

$$R_1{}^2 = R_2{}^2 = R_3{}^3 = (R_2R_3)^2 = (R_1R_2)^3 = (R_1R_3)^5 = 1. \tag{3.1}$$

They can be chosen to be the reflections in the three diametral planes 36, 25 and 34 respectively (figure 3).

$$\begin{aligned} R_1 &= (15)(24)(ae)(bd) \\ R_2 &= (16)(34)(af)(ce), \\ R_3 &= (1f)(25)(a6)(be). \end{aligned} \tag{3.2}$$

The permutation $R_1R_2R_3 = (13245acbde)(6f)$ of order ten acts cyclically on the vertices of a Petrie polygon [8]. $(R_1R_2R_3)^5 = (1a)(2b)(3c)(4d)(5e)(6f)$ is the central inversion. The group can be restricted to the factor group A_5 by including $(R_1R_2R_3)^5 = 1$ as an extra generating relation. We then obtain generating relations for A_5 in the form

$$\begin{aligned} R_1{}^2 = R_2{}^2 = R_3{}^2 = (R_2R_3)^2 = (R_1R_2)^3 \\ = (R_1R_3)^5 = (R_1R_2R_3)^5 = 1. \end{aligned} \tag{3.3}$$

The generating relations for $L_2(5) \sim A_5$ given by Conway [3] are obtained immediately

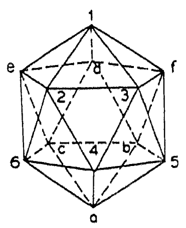

Figure 3.

by substituting

$$R_1 = \alpha\beta, \quad R_2 = \beta\gamma, \quad R_3 = \gamma. \tag{3.4}$$

That is, Conway's generators are given by

$$\alpha = R_1R_2R_3, \quad \beta = R_2R_3, \quad \gamma = R_3. \tag{3.5}$$

(The S, T and R of Todd [15] are respectively α, γ and $\beta\alpha$). The simplest set of generating relations for A_5 is that of the 'icosian calculus' of Hamilton [8], with only two generators ι and κ satisfying

$$\iota^2 = \kappa^3 = (\kappa\iota)^5 = 1. \tag{3.6}$$

They are obtained from (3.3) by substituting

$$\iota = R_2R_3, \quad \kappa = R_1R_2. \tag{3.7}$$

Employing (3.7), we find that $\kappa\iota\kappa^{-1}\iota = (R_1R_2R_3)^2$ and hence $(\kappa\iota\kappa^{-1}\iota)^3 = R_1R_2R_3$. We obtain then the inverse of the relations (3.7), in the form

$$\begin{aligned} R_1 &= (\kappa\iota\kappa^{-1}\iota)^3\iota = (\kappa\iota\kappa^{-1}\iota)^2\kappa\iota\kappa^{-1}, \\ R_2 &= (\kappa\iota\kappa^{-1}\iota)^3\iota\kappa = (\kappa\iota\kappa^{-1}\iota)^2\kappa\iota, \\ R_3 &= (\kappa\iota\kappa^{-1}\iota)^3\iota\kappa\iota = (\kappa\iota\kappa^{-1}\iota)^2\kappa. \end{aligned} \tag{3.8}$$

It is time-consuming, but not difficult, to deduce all the relations (3.3) from (3.6) and (3.8).

The rotation group A_5 of an icosahedron is the symmetry group of a hemi-icosahedron (icosahedron with opposite points identified). The hemi-icosahedron has 6 vertices, 15 edges and 10 faces. Half-turns about mid-points of edges are not distinguishable from reflections, and central inversion is the identity. In figure 4, the six vertices have been labelled by the six symbols of $\Omega(5)$. The permutations

$$R_1 = (04)(13), \quad R_2 = (0\,\infty)(23), \quad R_3 = (0\,\infty)(14) \tag{3.9}$$

are reflections in the three diametral planes σ_1, σ_2, σ_3 through the edges 2∞, 14 and 23 respectively. Alternatively (and equivalently), they are half-turns about the mid-points of the edges 04, $0\,\infty$ and 14 respectively. It is easily verified that α, β and γ of $L_2(5) \sim A_5$ are obtained when (3.9) is substituted into (3.5). The five synthemes (1.6) correspond to the five sets of mutually perpendicular golden rectangles inscribed in an icosahedron.

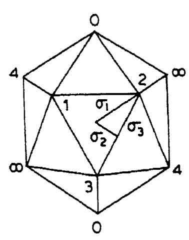

Figure 4.

r ten synthemes correspond to ten trios of golden rectangles like the one in Γhe ten trios are permuted by the rotations of the icosahedron.

hexads of the Steiner system $S(5, 6, 12)$ on $\Omega(11)$ are obtained from the generic 11) by applying the permutation α(1.15) repeatedly. These eleven hexads are l among themselves by the $L_2(11)$ generated by α, β and Δ. The eleven hexads ed as sets of labels for the vertices of eleven hemi-icosahedra. In figure 6, the sahedron of figure 4 has been relabelled by the symbols of $Q(11)$, according to spondence $\Omega(5) \to Q(11)$ given in (2.1).

l this hemi-icosahedron (0). Eleven hemi-icosahedra (i) ($i \in \Omega(11)$) are labelled e labelling of the vertices of $(i-1)$ is obtained from the labelling of the vertices ıdding one to each vertex label. We then find that, for any face of one of the sahedra, there is just one other hemi-icosahedron with a face labelled by the ee symbols. By identifying vertices that carry the same symbol, the hemi-ra fit together to form a polytope. This remarkable geometrical object was d by Grünbaum in 1976 [11] and independently by Coxeter in 1984 [9]. It has s, 55 edges, 55 faces and 11 cells. Its symmetry group is $L_2(11)$ and it is self-

ıbgroup A_5 that leaves the cell (0) invariant is generated by the three y permutations of $Q(11)$ induced by R_1, R_2 and R_3 under the correspondence 11):

$$S_1 = (14)(35)(27)(8X),$$

$$S_2 = (10)(27)(59)(6X),$$

$$S_3 = (26)(7X)(10)(34).$$

the permutations $\beta\Delta$, $\Delta^{\alpha^3}\Delta$ and $\Delta^{\alpha^{-3}}$. Observe that $S_1S_2S_3$ is β and S_2S_3 is Δ. ine S_4 to be reflection in the 145 face. This interchanges the two cells (0) and find

$$S_4 = (0X)(29)(36)(78).$$

ly verified, this is Δ^{α^4}. Finally, one can verify that $(S_1S_2S_3)^2\, S_1S_3S_4 = \alpha$. Thus $_3$ and S_4 generate $L_2(11)$.

lf-duality of the polytope corresponds to the automorphism θ of $L_2(11)$ (1.17). nutations α, β, Δ of the eleven vertices correspond, respectively, to the tions α^{-1}, β and Δ of the eleven cells.

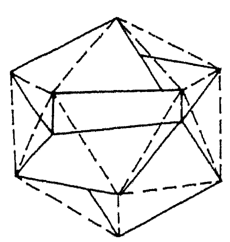

Figure 5.

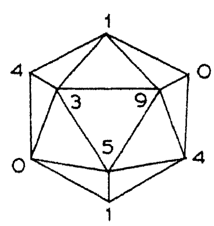

Figure 6.

4. The ternary Golay code

Let $\mathscr{V}$ be the vector space of 12-tuples over $GF(3)$, with a length measure imposed; the 12-tuples will be called *words* and the *weight* of a word is the number of non-zero components. The set $\Omega(11)$ can be employed as an index set for labelling the canonical base vectors (words of weight 1) $v_i\,(i\in\Omega)$. We find it convenient to let the index $i\in\Omega$ run through the values in the order

$$8\,2\,6\,7\,X\,0\,1\,3\,9\,5\,4\,\infty. \tag{4.1}$$

The *extended ternary Golay code* is the subspace $\mathscr{W}$ of $\mathscr{V}$ spanned by the rows $w_i\,(i\in\Omega)$ of the 12×12 matrix W defined by

$$W_{ij}=\begin{cases} 1, & i+j\in N \\ -1, & i+j\in Q \end{cases}. \tag{4.2}$$

With the canonical ordering (4.1) for the row and column labels,

$$W=\begin{bmatrix} X & j & Y & j \\ j^T & -1 & -j^T & 1 \\ Y & -j & -X & j \\ j^T & 1 & j^T & 1 \end{bmatrix}, \tag{4.3}$$

where j is a column of five 1s and X and Y are the 5×5 matrices $X=\mathrm{circ}(-1\,1\,-1\,-1\,1)$, $Y=\mathrm{circ}(-1\,-1\,1\,1\,-1)$ (i.e. the first rows of the matrices are as indicated and each row is obtained from the row above by cyclically shifting one place *to the right*). The elements of $\mathscr{W}$ (12-tuples consisting of linear combinations modulo 3 of the rows of W) are called *codewords.*

The code contains 3^6 words; the word of zero weight, 264 of weight 6, 440 of weight 9 and 24 of weight 12 ($1+264+440+24=729=3^6$). They can be characterized as the following particular linear combinations:

$\pm w_i$ are the 24 codewords of weight 12

$\pm(w_i+w_j\pm w_k)$ are the 440 codewords of weight 9

$\pm(w_i\pm w_j)$ are the 264 codewords of weight 6. (4.4)

The effect on the w_i of a change of basis in $\mathscr{V}$ is obtained by multiplication of W *on the right* by a nonsingular matrix. Multiplying by

$$S=\begin{bmatrix} -Y & j & & \\ j^T & 1 & & \\ & & Y & j \\ & & j^T & -1 \end{bmatrix} \tag{4.5}$$

gives

$$W'=\begin{bmatrix} & & -K & j \\ & I_6 & -j^T & 0 \\ K & j & I_6 & \\ -j^T & 0 & & \end{bmatrix} \tag{4.6}$$

where I_6 is the unit 6×6 matrix and $K = \text{circ}(0\ 1\ -1\ -1\ 1\ 1)$. We see immediately from the form of W' that $\mathscr{W}$ is 6-dimensional; denoting the ordered index sets $8267X0$ and 13954∞ by $\bar{N}$ and $\bar{Q}$ respectively, we see that either $w_i(i \in \bar{N})$ or $w_i(i \in \bar{Q})$ provide a basis for $\mathscr{W}$. The components of the second basis with respect to the first are given by the rows of $\begin{pmatrix} K & j \\ -j & 0 \end{pmatrix}$, i.e.:

$$\begin{array}{c} 1 \\ 3 \\ 9 \\ 5 \\ 4 \\ \infty \end{array} \begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix} \tag{4.7}$$

Of course, we have not preserved weights in going from W to W', so we cannot infer that the rows of W' span $\mathscr{W}$. On the other hand, by multiplication *on the left* by S, we get

$$U = \begin{bmatrix} & & K & -j \\ & I_6 & j^T & 0 \\ -K & -j & & I_6 \\ j^T & 0 & & \end{bmatrix}. \tag{4.8}$$

Either the first six rows of U, $u_i(i \in \bar{N})$, or the last six rows, $u_i(i \in \bar{Q})$, provide an alternative basis for the code.

The first of these is essentially the basis given originally by Golay [10].

The automorphism group of the Golay code consists of the linear transformations on $\mathscr{V}$ that preserve $\mathscr{W}$ and the weights of codewords. It is a binary Mathieu group $2M_{12}$ generated by the permutations

$$\begin{aligned} &A: v_i \to v_{i\alpha}, \quad B: v_i \to v_{i\beta}, \\ &C: v_i \to \pm v_{i\gamma}, \quad D: v_i \to v_{i\Delta}, \end{aligned} \tag{4.9}$$

of the 24 words of length 1, $\pm v_i$. The sign in C is $+$ for $i \in Q$ and $-$ for $i \in N$. The effect on the $w_i(i \in \Omega)$ is

$$\begin{aligned} &A: w_i \to w_{i\alpha}, \quad B: w_i \to w_{i\beta}, \\ &C: w_i \to \mp w_{i\gamma}, \quad D: w_i \to w_{i\Delta}, \end{aligned} \tag{4.10}$$

(where the sign in C is $-$ for $i \in Q$ and $+$ for $i \in N$). This remarkable result is due to Conway [3].

We now interpret the words projectively, as sets of homogeneous coordinates of points in a $PG(11, 3)$, which we call $\bar{\mathscr{V}}$. The code words are the sets of homogeneous coordinates of a subspace $PG(5, 3)$ which we call $\bar{\mathscr{W}}$. The twelve points $w_i(i \in \Omega)$ are thus twelve points in $PG(5, 3)$ whose coordinates are given by the rows of I_6 and the rows of the matrix (4.7). They are thus just the 12 points of Coxeter's configuration, which was shown by Coxeter[6] to have an automorphism group M_{12} consisting of collineations.

The correspondence between our notation and Coxeter's is:

$$\begin{matrix} w_8 & w_2 & w_6 & w_7 & w_X & w_0 & w_1 & w_3 & w_9 & w_3 & w_4 & w_\infty \\ 1 & 2 & 3 & 4 & 5 & f & a & b & c & d & e & 6 \end{matrix}. \tag{4.11}$$

Interpreted as collineations in $PG(5,3)$, the transformations (4.10) are the collineations given by the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.12}$$

(Check: multiply I_6 and the matrix (4.7) on the right!). They generate the group M_{12} of collineations that permute the 12 points.

The matrix W is symmetric and satisfies $W^2 = 0$. It follows immediately that $\mathscr{W}$ lies entirely in the quadric $x_i x_i = 0$ $(i \in \Omega)$ in $\mathscr{V}$ and that $\mathscr{W}$ coincides with its polar 5-space. Thus, a point of $\mathscr{V}$ is in $\mathscr{W}$ if and only if it lies in the polar primes of all the twelve points $w_i (i \in \Omega)$. Therefore, a word with components $c_i (i \in \Omega)$ is a codeword if and only if $c_i w_i = 0$. This establishes a one-to-one correspondence between points in $\mathscr{W}$ and syzygies satisfied by the twelve points (a syzygy is an equation expressing the vanishing of a linear combination of the w_i). Since there are 264 codewords of weight 6, there will be $264/2 = 132$ syzygies of length 6 satisfied by the coordinate of the twelve points of w_i. Coxeter lists them all explicitly. Each syzygy of length 6 expresses the condition that 6 points lie in a 4-space. Thus, the 12 points lie in sixes on 132 primes (hyperplanes) of the $PG(5,3)$. Each hexad of points is one of the hexads of the Steiner system $S(5,6,12)$.

It is useful at this stage to give a brief outline of the results of Coxeter's investigation of the configuration of 12 points in $PG(5,3)$.

A *hexastigm* is a set of six general points in a projective 4-space. A pair of *Möbius simplexes* in a projective space is a pair with the property that each vertex of one lies in a prime face of the other. The twelve points in $PG(5,3)$ can be regarded as a pair of complementary hexastigms in 66 ways, or as a pair of Möbius simplexes in 395 ways. The automorphism group M_{12} can be generated by 395 involutory collineations, each of which interchanges the simplexes of a Möbius pair. The coordinates of the points $ABCDEF$ of a hexastigm can be chosen (by adjusting irrelevant factors) to satisfy a syzygy

$$A+B+C+D+E+F=0. \tag{4.13}$$

One can then define 40 associated points as follows:
15 'harmonic' or negative points such as $A-B$ etc. 15 'positive' points such as $A+B-C-D\,(=C+D-E-F=E+F-A-B)$ etc. 10 'minor' points such as $A+B+C$ $(=-D-E-F)$ etc. If the underlying field of the projective space in which the hexastigm lies is $GF(3)$, the 40 associated points all lie in a 3-space (since they are all obtainable as linear combinations of, for example, $A-F$, $B-F$, $C-F$ and $D-F$). Moreover, they are *all* of the points of the 3-space in which they lie $(40=(3^4-1)/2)$. The negative points can be denoted by *duads* such as (AB) and the positive points by synthemes such as $(AB.\ CD.\ EF)$. The minor points may be denoted by $P_{ABC}\,(=P_{DEF})$ etc. The associated 3-spaces of a pair of complementary hexastigms coincide. More precisely, the negative points of each coincide with the positive points of the other, and the minor points of one coincide with minor points of the other. Each of these coincidences is indicated by several of the 132 syzygies. For example, consider Coxeter's two generic hexastigms 123456 and *abcdef* (in our notation, $w_i(i\in N)$ and $w_i(i\in Q)$). The coincidence $(16)=(af.\,cd.\,eb)$ is seen in the three syzygies

$6+c+d=1+e+b$, $6+e+b=1+a+f$, $6+a+f=1+c+d$ of Coxeter's list, and $P_{123}=P_{aef}$ is seen in the four syzygies $1+2+3+a+e+f=0$, $4+5+6=a+e+f$, $1+2+3=b+c+d$, $4+5+6+b+c+d=0$. The two generic syzygies $1+2+3+4+5+6=0$, $a+b+c+d+e+f=0$ together with all the syzygies associated with the coincidences of associated points, account for the whole list; $2+4{\cdot}10+3{\cdot}15+3{\cdot}15=132$.

The coincidences between negative points of 123456 and positive points of *abcdef* establish a one-to-one correspondence between number duads and letter synthemes. The table of duads and synthemes obtained from this one by the permutation $(1a)\,(2b)\,(3c)\,(4d)\,(5e)\,(6f)$ lists the coincidences between negative points of 123456 and positive points of *abcdef*. The permutation interchanges these two hexastigms and also the two simplexes $12345f$, $abcde6$ of a Möbius pair. It corresponds to an involutary collineation in M_{12}. In our notation, it is the collineation with matrix $(CD)^5$ (see (2.4) and (4.11)). This matrix is (4.7) with the sign of the final row changed.

Of the $(3^6-1)/2=364$ points of $PG(5,3)$, twelve are the points w_i, 132 occur as positive or negative points and 220 occur as minor points $(12+132+220=364)$. The positive or negative points are the $w_i\pm w_j$ which arise from the codewords of weight 6 when the Golay code is interpreted projectively, and the minor points are the $w_i+w_j\pm w_k$ which arise from the codewords of weight 9. The twelve points w_i of course arise from the words of weight 12.

The words of weight 1 can be interpreted projectively as the coordinates of twelve primes in $\mathscr{V}$ (prime faces of the reference simplex). We call these twelve primes $\pi_i(i\in\Omega)$. Consider the intersection of the primes π_i by the 5-space $\mathscr{W}$. We get twelve primes in $\mathscr{W}$, which will also be called π_i. Referred to the coordinate system that we have established in $\mathscr{W}$ (with $w_i(i\in\bar{N})$ as reference simplex), the coordinates of the twelve primes $\pi_i(i\in\Omega)$ are given (in the order (4.1)) by the *columns* of the matrices

$$\begin{bmatrix} X & j \\ j^T & -1 \end{bmatrix}, \quad \begin{bmatrix} Y & j \\ -j^T & 1 \end{bmatrix} \tag{4.14}$$

(Proof: Let c be a sextuple of homogeneous coordinates of a point in $\bar{\mathscr{W}}$. The components of this point, referred to the coordinate system of $\bar{\mathscr{V}}$, are given by the 12-tuple obtained by multiplying the row c by

$$\begin{pmatrix} X & j & Y & j \\ j^T & -1 & -j^T & 1 \end{pmatrix},$$

since this array is just the upper half of W and gives the coordinates of the base points $w_i(i \in \bar{N})$, of $\bar{\mathscr{W}}$. The condition for a point to lie on π_i is that the ith component of its 12-tuple shall vanish. Therefore, the prime π_i of $\bar{\mathscr{W}}$ is given by the ith column of the above array).

Clearly, the 12 primes π_i of $\bar{\mathscr{W}}$ are the 12 primes introduced into Coxeter's configuration by Todd [16]. The correspondence between our notation and Todd's is the following:

$$\frac{\pi_8 \quad \pi_2 \quad \pi_6 \quad \pi_7 \quad \pi_X \quad \pi_0 \quad \pi_1 \quad \pi_3 \quad \pi_9 \quad \pi_5 \quad \pi_4 \quad \pi_\infty}{\mathbf{a} \quad \mathbf{b} \quad \mathbf{c} \quad \mathbf{d} \quad \mathbf{e} \quad \mathbf{-6} \quad \mathbf{-1} \quad \mathbf{-2} \quad \mathbf{-3} \quad \mathbf{-4} \quad \mathbf{-5} \quad \boldsymbol{f}}. \qquad (4.15)$$

Multiplying the two matrices (4.13) on the left by $M^{-1} = \begin{pmatrix} X & j \\ j^T & -1 \end{pmatrix}$ transforms them to $\begin{pmatrix} K & j \\ j^T & 0 \end{pmatrix}$ and $\begin{pmatrix} -I_5 & 0 \\ 0 & 1 \end{pmatrix}$ respectively. The columns of these matrices, taken in order, are Coxeter's coordinate sextets $a, b, c, d, e, -6, -1, -2, -3, -4, -5, f$. Thus, the twelve primes are the polars of the twelve points, with respect to the quadric whose matrix is $M = \begin{pmatrix} -Y & j \\ j^T & 1 \end{pmatrix}$. The polars of Coxeter's 123456*abcdef* are, respectively, Todd's **123456abcdef.** There are 95040 quadrics with the property that the twelve primes are the polars of the twelve points (obtained from M by multiplying on the right by the transpose of any matrix of the collineation group M_{12}). A different quadric, more suitable to our present notation, is the quadric with matrix $Q = \begin{pmatrix} Y & j \\ -j^T & 1 \end{pmatrix}$ Multiplying the matrices (4.7) on the left by $Q^{-1} = \begin{pmatrix} -X & -j \\ -j^T & -1 \end{pmatrix}$ gives, respectively, I_6 and the transpose of the matrix (4.7). Thus, with respect to this quadric, the polarity is $\pi_i \leftrightarrow w_i(i \in \Omega)$.

The polarities show that the configuration of 12 points and 12 primes in $PG(5,3)$ is self-dual. One can immediately infer the dual of all the properties of Coxeter's configuration: the 12 primes pass in sixes through 132 points and can be regarded in 395 ways as a pair of Möbius simplexes. Given any 'dual hexastigm' (six primes through a point), one can construct 40 associated primes: 15 'negative' primes, 15 'positive' primes and 10 'minor' primes. The associated primes of two complementary dual hexastigms coincide.

The 364 primes of $PG(5,3)$ can be classified as follows. Twelve of them are the π_i, arising from the words of weight 1 of $\mathscr{V}$. 132 of them occur as 'positive or negative' primes; these arise from the words of weight 2, $\pi_i \pm \pi_j$. And 220 of them occur as 'minor' primes; these arise from the words of weight 3, $\pi_i + \pi_j \pm \pi_k$.

In $\mathscr{V}$, the words of weight m have $12-m$ zero components. Hence, in $\bar{\mathscr{V}}$, a point whose coordinate 12-tuple is a word of weight m lies on just $12-m$ of the primes π_i. In particular, in $\bar{\mathscr{W}}$ the twelve points w_i lie on none of the twelve primes π_i, the 220 'minor' points each lie on just 3 of the primes π_i, and the 132 'positive and negative' points each lie on just 6 of the primes π_i. The dual of this latter statement is: the 132 'positive and negative' primes each contain six of the twelve points w_i. Thus *the primes in which Coxeter's* 132 *hexastigms lie are the 'positive and negative' primes* $\pi_i \pm \pi_j$. In fact, the two primes $\pi_i \pm \pi_j$ (for fixed i and j) are the primes of a complementary pair of hexads.

Given any hexastigm of the points w_i, one can associate with it, according to the above, two of the primes α_i. We get 132 *octads* consisting of six of the points and two of the primes. Dually we get 132 octads consisting of two of the points and six of the primes. We can also form 495 octads consisting of four points and four primes, as follows. There are $\binom{12}{4} = 495$ ways of choosing four of the twelve primes. Going over to the $PG(11,3)$ in which these four primes are specified by words of weight one, we can see by studying the rows of W that there are just four 'positive and negative' points that lie on all four of the primes. For example, π_8, π_2, π_6 and π_7 all contain $w_2 + w_6, w_x + w_4, w_0 - w_\infty, w_1 + w_5$. Just four of the symbols w_i have not appeared, namely w_8, w_7, w_3 and w_4. So we can specify an octad $\pi_8\pi_2\pi_6\pi_7 w_8 w_7 w_3 w_4$. The total number of octads is now $132 + 132 + 495 = 759$. Todd [16] shows that these octads are the octads of a Steiner system $S(5, 8, 24)$ on the 24 objects π_i, w_i.

5. The binary Golay code

Let $\mathscr{V}$ now denote a vector space of 24-tuples over $GF(2)$ with a length measure imposed on it; the *weight* of a 24-tuple is the number of non-zero elements. The 24-tuples will be called *words*. The set $\Omega(23)$ can be used as an index set labelling the base vectors v_i $(i \in \Omega)$ and we let the index i run through Ω in the order

$$\begin{matrix} 5 & 10 & 20 & 17 & 11 & 22 & 21 & 19 & 15 & 7 & 14 & 0 \\ 1 & 2 & 4 & 8 & 16 & 9 & 18 & 13 & 3 & 6 & 12 & \infty \end{matrix} \tag{5.1}$$

(5 times the powers of 2 modulo 23, in order, followed by zero, followed by the powers of 2, and finally the symbol ∞). We shall call the ordered set consisting of the first twelve of these symbols $\bar{N}$, and the ordered set consisting of the last twelve will be called $\bar{Q}$. The *extended binary Golay code* is the subspace $\mathscr{C}$ spanned by the rows N_i $(i \in \Omega)$ of the matrix N defined by

$$N_{ij} = \begin{cases} 1, & i + j \in N(23) \\ 0, & i + j \in Q(23). \end{cases} \tag{5.2}$$

With the canonical ordering of the row and column labels,

$$N = \begin{bmatrix} X & j & Y & j \\ j^T & 0 & 0 & 1 \\ Y^T & 0 & Z & j \\ j^T & 1 & j^T & 1 \end{bmatrix} \tag{5.3}$$

where j is a column of 1s, 0 is the zero element or a row or column of eleven 0s, depending on the context, and X, Y and Z are the 11×11 matrices

$$\begin{aligned} X &= \text{circ}\,(11010000101), \\ Y &= \text{circ}\,(01001100011), \\ Z &= \text{circ}\,(00101111010). \end{aligned} \tag{5.4}$$

The 12-tuples of $\mathscr{C}$ are called *codewords*. There are 2^{12} codewords; one of weight zero, 759 of length 8, 2576 of weight 12, 759 of weight 16 and 1 of weight 24 $(1 + 759 + 2576 + 759 + 1 = 4096 = 2^{12})$. Observe that the rows of the matrix N are 23 words of weight 12 and one of weight 24 (N_∞). Thus there is a fundamental asymmetry that was not present in the case of the ternary code—the rows of W were all of the same weight. This distinction turns out to be crucially important.

Changing the basis in $\mathscr{V}$ multiplies N *on the right* by a non-singular matrix. Multiplying by

$$S = \begin{bmatrix} P & j & & \\ j^T & 1 & & \\ & & Q & j \\ & & j^T & 0 \end{bmatrix}, \tag{5.5}$$

where $P = \text{circ}\,(01100000011)$ and $Q = \text{circ}\,(10011111100)$, gives

$$N' = \begin{bmatrix} & I_{12} & L & j \\ & & j^T & 0 \\ L^T & j & I_{12} & \\ j^T & 0 & & \end{bmatrix} \tag{5.6}$$

where

$$L = \text{circ}\,(111011100010). \tag{5.7}$$

We see from the form of N' that $\mathscr{C}$ is 12-dimensional, and that either the first twelve rows of $N(N_i,\, i \in \bar{N})$ or the last twelve rows $(N_i,\, i \in \bar{Q})$ provide a basis for $\mathscr{C}$.

An alternative pair of bases for $\mathscr{C}$ is obtained by multiplying N *on the left* by S. We get the same matrix N'(5.5), but the interpretation is different. We establish in this way that either the first twelve rows of N' or the last twelve rows of N' provide a basis for $\mathscr{C}$ (this was *not* established by the previous derivation of N'!). We shall call the rows of N', $u_i(i \in \Omega)$. The basis $u_i(i \in \Omega)$, consisting of the rows of the array

$$\begin{pmatrix} I_{12} & L & j \\ & j^T & 0 \end{pmatrix} \tag{5.8}$$

is the usual basis for the code. This elegant characterization of the code was discovered by Karlin [12]. Golay's original characterization [10] had a different matrix in place of L, related to L by permutations of the rows and columns. It is remarkable that the circulant L in Karlin's construction of the code is related to the quadratic residues modulo 11:The first row of L, (11011100010), is obtained from the set $\Omega'(11) = 0123456789X$ by replacing each symbol by 0 if it is in $N(11)$ and by 1 if it in $Q(11)$. The array (5.7) is essentially the array employed by Leech [13] in the construction of his celebrated lattice [18].

Omitting the null word, $\mathscr{C}$ can be interpreted projectively as a subspace $PG(11,2)$ (which we call $\bar{\mathscr{C}}$) of a $PG(23,2)$ (which we call $\bar{\mathscr{V}}$). Choosing the points $N_i(i\in\bar{Q})$ as the vertices of the reference simplex in $\bar{\mathscr{C}}$, the homogeneous coordinates of the other twelve points $N_i(i\in\bar{N})$ are given by the rows of $\begin{pmatrix} L & j \\ j^T & 0 \end{pmatrix}$, i.e.

$$\begin{array}{r} 5 \\ 10 \\ 20 \\ 17 \\ 11 \\ 22 \\ 21 \\ 19 \\ 15 \\ 7 \\ 14 \\ 0 \end{array} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad (5.9)$$

The 24 points $N_i(i\in\Omega)$ are the 24 points in $PG(11,2)$ studied by Todd [16] [17], which are permuted by a group M_{24} of collineations.

The automorphism group of the binary Golay code is an M_{24} of permutations of the words of length 1. It is generated by the three permutations

$$A{:}v_i\to v_{i\alpha},\quad B{:}v_i\to v_{i\beta},\quad H{:}v_i\to v_{i\gamma\delta}, \qquad (5.10)$$

where α, β and $\gamma\delta$ are the permutations of $\Omega(23)$ given in (1.19) and (1.21). The effect of A and B on the $N_i(i\in\Omega)$ is found to be given by the permutations

$$A{:}N_i\to N_{i\alpha^{-1}},\quad B{:}N_i\to N_{i\beta} \qquad (5.11)$$

but H does not permute them. The word of length 24, N_∞, remains fixed, but the other 23 are mapped to 23 different words. The images of the N_i are, however, still in the subspace $\bar{\mathscr{C}}$.

When this M_{24} is interpreted as a group of collineations in $\bar{\mathscr{C}}$, we find that A, B and H are the collineations with matrices given by

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B=\begin{bmatrix}
0 & 1 & & & & & & & & & & \\
0 & 0 & 1 & & & & & & & & & \\
0 & 0 & 0 & 1 & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

$$H=\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \qquad (5.12)$$

That the collineations A and B have the correct action (5.11) on the 24 points $N_i(i\in\Omega)$ is easily checked: the coordinates of these 24 points of $\mathscr{C}$ are the rows of (5.9) and the unit matrix, so their images under the action of A and B are given by the rows of the matrices given by multiplying (5.9) and I_{12} on the right of the matrices A and B.

The matrix H was obtained as follows: from the matrix N (5.9) we see that $N_1 = v\{10\,20\,21\,19\,14\,4\,16\,9\,18\,13\,6\,\infty\}$ (the notation vS means $\sum v_i(i\in S)$, where S is any subset of Ω). Acting on this with the transformation H given by (5.10) gives the image of N_1, which is $v\{13\,16\,18\,14\,11\,7\,21\,17\,5\,22\,0\}$. In terms of the basis $u_i(i\in\bar{Q})$ given by the last six rows of N' (5.6), this must be $u\,\{13\,16\,8\,1\,4\}$ (which can be verified by actually adding these six u_i, thus establishing that the image of N_1 really is in $\mathscr{C}$). Finally we go over to the basis N_i $(i\in\bar{Q})$ which is related to the basis $u_i(i\in\bar{Q})$ by the matrix $\begin{pmatrix} Q & j \\ j^T & 0 \end{pmatrix}$ appearing in (5.5). We find $u\,\{13\,16\,8\,1\,4\} = N\{8\,16\,18\,13\,6\,\infty\}$. This establishes the first row of the matrix H. (The process sounds complicated when described but in practice is quite rapid).

Observe the curious structure of the matrix H. The 11×11 submatrix obtained by omitting the final row and final column has the property that each row is obtained from the previous row by cyclically shifting three places to the left. The explanation for this property is as follows. The α and β of M_{12} satisfy the generating relation $\alpha^\beta = \alpha^3$ [3].

Recall that they correspond to the β and δ of M_{24} under the embedding of M_{12} in M_{24} discussed in § 2. Therefore, the β and δ of M_{24} satisfy $\beta^{\delta} = \beta^3$. Together with the M_{24} generating relation $(\beta\gamma)^2 = 1$, this gives $\beta(\gamma\delta) = (\gamma\delta)\beta^{-3}$. Hence $BH = HB^{-3}$, which is the structural property of H that we noticed.

Since the point N_∞ remains invariant under the group M_{24} of collineations, the 12-dimensional representation of M_{24} generated by A, B and H is *reducible*. An *eleven-dimensional representation* of M_{24} is obtained by removing the final rows and columns from the matrices A, B and H. (In the projective geometry of $\bar{\mathscr{C}}$ this corresponds to projection from the point N_∞ (00...01) to the prime [00...01].

Presumably, the 11-dimensional representation of M_{23} discovered by Paige [14] is equivalent to the one generated by the reduced matrices A and B together with a matrix D obtained by omitting the last row and column from H^6. These three matrices represent the generators α, β, δ of the M_{23} that fixes v_∞. We find

$$D = \begin{bmatrix} 0&0&1&0&1&0&1&0&1&0&1 \\ 1&0&1&0&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&0&1&0&1 \\ 1&0&1&0&1&0&1&0&1&0&0 \\ 1&0&0&1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&0&1&0&1&0&1 \\ 1&0&1&0&1&0&1&0&0&1&0 \\ 0&1&0&1&0&1&0&1&0&1&0 \\ 0&1&0&0&1&0&1&0&1&0&1 \\ 1&0&1&0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1&0&0&1 \end{bmatrix} \tag{5.14}$$

A different 12-dimensional representation of M_{24} is obtained by considering the collineations on $\bar{\mathscr{C}}$ which permute the 24 points $N_i (i \in \Omega)$: This is generated by the A and B of (5.12) together with

$$F: N_i \to N_{i\gamma\delta}. \tag{5.15}$$

The matrix of the collineation F on $\bar{\mathscr{C}}$ is

$$F = \begin{bmatrix} 1&0&1&1&1&0&0&0&1&0&1&1 \\ 1&1&0&0&0&1&0&1&1&0&1&1 \\ 0&0&1&0&1&1&0&1&1&1&0&1 \\ 0&1&1&0&1&1&1&0&0&0&1&1 \\ 0&1&1&1&0&0&0&1&0&1&1&1 \\ 1&0&0&0&1&0&1&1&0&1&1&1 \\ 0&1&0&1&1&0&1&1&1&0&0&1 \\ 1&1&0&1&1&1&0&0&0&1&0&1 \\ 1&1&1&0&0&0&1&0&1&1&0&1 \\ 0&0&0&1&0&1&1&0&1&1&1&1 \\ 1&0&1&1&0&1&1&1&0&0&0&1 \\ 1&1&1&1&1&1&1&1&1&1&1&0 \end{bmatrix} \tag{5.16}$$

as is easily verified by multiplying I_{12} and the matrix (5.9) on the right.

6. Octastigms and dodecastigms

An *octastigm* may be defined as a set of eight general points in a projective 6-space. The coordinate sets of the eight points of an octastigm $ABCDEFGH$ can be chosen to satisfy a syzygy

$$A+B+C+D+E+F+G+H=0. \tag{6.1}$$

Associated with an octastigm are $\binom{8}{2}=28$ *duadic* points $A+B$ etc., and $\frac{1}{2}\binom{8}{4}=35$ *tetradic* points $A+B+C+D$ etc. In a projective space over $GF(2)$, these points associated with an octastigm all lie in a $PG(5,2)$ (their coordinate sets can all be obtained as linear combinations of those of six of the duadic points such as $A+H$, $B+H, C+H, D+H, E+H, F+H$); indeed they are all the points of the $PG(5,2)$ in which they lie $(28+35=63=2^6-1)$.

The tetradic points associated with an octastigm in a projective space over $GF(2)$ lie in sevens on thirty planes. For example, the seven points

$$\begin{aligned}
A+D+G+H &= B+C+E+F,\\
B+E+G+H &= C+A+F+D,\\
C+F+G+H &= A+B+D+E,\\
D+B+C+H &= A+E+F+G,\\
E+C+A+H &= B+F+D+G,\\
F+A+B+H &= C+D+E+G,\\
D+E+F+H &= A+B+C+G,
\end{aligned} \tag{6.2}$$

all lie in a $PG(2,2)$ (they are all given by linear combinations of, for example, $A+D+G+H$, $B+E+G+H$ and $F+A+B+H$). The Fano 7_3 whose vertices are the seven points (6.2) is indicated in figure 7.

A *dodecastigm* is a set of twelve general points in a projective 10-space. The homogenous coordinate sets of a dodecastigm $ABCDEFGHIJKL$ can be made to satisfy a syzygy

$$A+B+C+D+E+F+G+H+I+J+K+L=0. \tag{6.3}$$

Associated with a dodecastigm are $\binom{12}{2}=66$ *duadic* points, $\binom{12}{4}=495$ *tetradic* points and $\frac{1}{2}\binom{12}{6}=462$ *hexadic* points. In a projective space over $GF(2)$, these

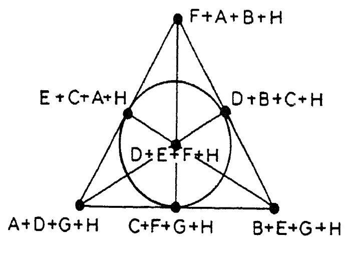

Figure 7.

associated points all lie in a 9-space, and are *all* the points of the $PG(9,2)$ in which they lie $(66+495+462=1023=2^{10}-1)$.

The set of 24 points $N_i(i \in \Omega)$ in $PG(11,2)$ constitute 2576 dodecastigms (in complementary pairs). The 63 associated points of a dodecastigm coincide with the 63 associated points of the complementry dodecastigm. The situation is analogous to that of the 12 points in $PG(5,3)$, which constitute 132 hexastigms, the associated (positive, negative and minor) points of a hexastigm coinciding with those of the complementary hexastigm.

Moreover, the 24 points constitute 759 octastigms, corresponding to the octads of a Steiner system $S(5,8,24)$. The dodecastigms correspond to the umbral dodecads of the Steiner system (Todd). The automorphism group of the Steiner system is realized as the group M_{24} of collineations generated by A, B (5.12) and F (5.16).

A set of three octastigms which together contain all 24 points corresponds to a *trio* of the Steiner system. The three octastigms of a trio have seven of their tetradic points in common—the seven points of the unique $PG(2,2)$ common to the three 6-spaces of the octastigms. The configuration of 24 points in $PG(11,2)$ can be regarded as a trio of octastigms in 3795 ways.

In order to establish the above facts, recall that the coordinatisation of $\mathscr{C}$ can be chosen so that the $N_i(i \in \bar{Q})$ are given by the rows of the unit 12×12 matrix and the $N_i(i \in \bar{N})$ by the rows of (5.9). We thus get eleven syzygies of length 8, corresponding to eleven octastigms, each expressing the coincidence of a duadic point of the generic dodecastigm $N_i(i \in N)$ with a hexadic point of the generic dodecastigm $N_i(i \in Q)$. They are obtained by applying the powers of the transformation $B: N_i \rightarrow N_{2i}$ to the syzygy

$$N\{5\infty\} = N\{1 \quad 2 \quad 8 \quad 16 \quad 9 \quad 6\}. \tag{6.4}$$

We call these *syzygies of type* (2, 6). The final row of (5.9) provides the syzygy

$$N\{0 \quad 1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 9 \quad 18 \quad 13 \quad 3 \quad 6 \quad 12\} = 0 \tag{6.5}$$

of length 12 and type (0, 12), corresponding to the generic dodecastigm $N_i(i \in Q)$. (The notation NS, $S \in \Omega$ denotes $\Sigma N_i(i \in S)$). Adding these twelve syzygies in pairs gives $\binom{12}{2}$ = 132 syzygies of type (2, 6). Each indicates an octastigm and at the same time indicates a coincidence between a duadic point of $N_i(i \in N)$ and a hexadic point of $N_i(i \in Q)$. Adding them in fours gives $\binom{12}{4} = 495$ syzygies of type (4, 8). Similarly, by choosing $N_i(i \in \bar{N})$ as the reference points, we get eleven syzygies of type (6, 2), obtained from

$$N\{1 \quad 0\} = N\{5 \quad 20 \quad 21 \quad 19 \quad 15 \quad 14\} \tag{6.6}$$

by applying the powers of B, and one of type (12, 0), namely

$$N\{5 \quad 10 \quad 20 \quad 17 \quad 11 \quad 22 \quad 21 \quad 19 \quad 15 \quad 7 \quad 14\} = 0. \tag{6.7}$$

Adding these twelve syzygies in twos gives 132 of type (6, 2) and adding them in fours gives 495 of type (8, 4).

495 syzygies of type (4, 4) are obtained either by adding (6.5) to those of type (4, 8) or by adding (6.7) to those of type (8, 4). We obtain the number of octastigms,

$$132 + 132 + 495 = 759. \tag{6.8}$$

They constitute the octads of a Steiner system $S(5,8,24)$. (The proof of this, which consists of showing that any five of the points belong to a unique octad, has been given by Todd [16]. The automorphism group of the $S(5,8,24)$ is of course the M_{24} of collineations that permutes the N_i, generated by A, B and F.

We have seen that 66 of the hexadic points of $N_i(i \in Q)$ coincide with duadic points of $N_i(i \in N)$. The remaining 396 hexadic points of $N_i(i \in Q)$ coincide with hexadic points of $N_i(i \in N)$. This provides 396 syzygies of type (6, 6). By adding (6.5) and (6.7) to these we get altogether $4 \times 396 = 1584$ of type (6, 6) (they emerge upon adding those of type (6, 2) to those of type (2, 6)). We now have all the syzygies of length 12: one of type (0, 12), one of type (12, 0), 495 each of types (4, 8) and (8, 4) and 1584 of type (6, 6). The total number of dodecastigms is

$$1 + 1 + 495 + 495 + 1584 = 2576.$$

They are the umbral dodecads of the Steiner system $S(5,8,24)$. Given a complementary pair of dodecastigms (duum of the Steiner system) their tetradic points coincide, the duadic points of one coincide with the hexadic points of the other, and the remaining hexadic points coincide. Those hexadic points of one of the dodecastigms that coincide with duadic points of the other are just the hexads of the Steiner system of the M_{12} of collineations that preserves the dodecastigm pair.

Finally, consider a generic trio of octastigms with no common point. We can take them to be the three generic octads given in (2.6), which have the syzygies

$$\begin{aligned} N\{0 \quad 8 \quad 20 \quad 14 \quad 15 \quad 3 \quad 18 \quad \infty\} &= 0, \\ N\{4 \quad 13 \quad 7 \quad 11 \quad 10 \quad 16 \quad 2 \quad 17\} &= 0, \\ N\{9 \quad 1 \quad 12 \quad 22 \quad 6 \quad 5 \quad 21 \quad 19\} &= 0. \end{aligned} \tag{6.9}$$

The $L_2(7)$ of section 2 induces the group $L_2(7)$ of collineations generated by

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 1&0&1&1&0&1&1&1&0&0&0&1\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&1&1&0\\ 0&1&1&1&0&0&0&1&0&1&1&1\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 1&1&0&1&1&1&0&0&0&1&0&1\\ 0&0&0&1&0&1&1&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&1&1&0&1&1&1&0&0&0&1\\
0&0&1&0&1&1&0&1&1&1&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&1&1&1&0&0&0&1&0&1&1&1\\
1&0&1&1&1&0&0&0&1&0&1&1\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix},$$

$$\begin{bmatrix}
1&0&0&0&1&0&1&1&0&1&1&1\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&1&0&1&1&0&1&1&1&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&1&1&0&1&1&1&0&0&0&1&1\\
1&1&0&0&0&1&0&1&1&0&1&1\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&0&1&1&0&1\\
1&1&0&1&1&1&0&0&0&1&0&1\\
0&0&0&1&0&1&1&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&0
\end{bmatrix} \qquad (6.10)$$

(induced by the α, β and γ of $L_2(7)$). That they have the correct action on the N_i can be checked by multiplying I_{12} and the matrix (5.9) on the right.

The seven objects P_i(1.12) constructed from $PL(7)$ correspond, under the correspondences (2.6) to seven tetradic points that the three octastigms have in common:

$$\begin{array}{llll}
P_0\colon N\{0\ \ 8\ \ 20\ \ 15\} & = N\{4\ \ 13\ \ 7\ \ 16\} & = N\{9\ \ 1\ \ 12\ \ 6\} \\
P_1\colon N\{8\ \ 20\ \ 14\ \ 3\} & = N\{13\ \ 7\ \ 11\ \ 16\} & = N\{1\ \ 12\ \ 22\ \ 5\} \\
P_2\colon N\{20\ \ 14\ \ 15\ \ 18\} & = N\{7\ \ 11\ \ 10\ \ 2\} & = N\{12\ \ 22\ \ 6\ \ 21\} \\
P_3\colon N\{14\ \ 15\ \ 3\ \ 0\} & = N\{11\ \ 10\ \ 16\ \ 4\} & = N\{22\ \ 6\ \ 5\ \ 9\} \\
P_4\colon N\{15\ \ 3\ \ 18\ \ 8\} & = N\{10\ \ 16\ \ 2\ \ 13\} & = N\{6\ \ 5\ \ 21\ \ 1\} \\
P_5\colon N\{3\ \ 18\ \ 0\ \ 20\} & = N\{16\ \ 2\ \ 4\ \ 7\} & = N\{5\ \ 21\ \ 9\ \ 12\} \\
P_6\colon N\{18\ \ 0\ \ 8\ \ 14\} & = N\{2\ \ 4\ \ 13\ \ 11\} & = N\{21\ \ 9\ \ 1\ \ 22\}.
\end{array} \qquad (6.15)$$

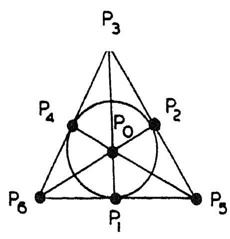

Figure 8.

The coincidences indicated by the equality signs correspond to various other octastigms. That these actually are octads of the $S(5,8,24)$ is readily visible in the MOG. That these seven points P_i of $PG(11,2)$ are collinear in threes (for example $P_0 + P_1 + P_3 = 0$) is easily checked. They form a Fano configuration (figure 8) and therefore constitute a $PG(2,2)$.

7. Twenty-four primes in *PG*(11, 12)

The words of unit weight in $\mathscr{V}$ are the sets of homogeneous coordinates for the 24 reference primes in V. We call these primes $\pi_i(i \in \Omega)$. In the canonical order (5.1) they are given by the columns (or rows) of the unit matrix I_{24}. They intersect $\mathscr{C}$ in twelve primes of $\mathscr{C}$, which we shall also call $\pi_i(i \in \Omega)$. From the form of the matrix N, (5.3), we see that there is just one of them, π_∞, that contains none of the 24 points $N_i(i \in \Omega)$ and that there is just one of the 24 points, N_∞ that is not contained in any of the 24 primes. The remaining 23 points and 23 primes constitute a configuration 23_{12} in $\mathscr{C}$.

In the coordinate system on $\mathscr{C}$ in which $N_i(i \in \bar{N})$ and $N_i(i \in \bar{Q})$ are given respectively by the rows of the matrix (5.9) and the rows of I_{12}, the primes $\pi_i(i \in \bar{N})$ and the primes $\pi_i(i \in \bar{Q})$ are given respectively by the columns of the two 12×12 matrices

$$\begin{bmatrix} Y^T & 0 \\ j^T & 1 \end{bmatrix}, \quad \begin{bmatrix} Z & j \\ j^T & 1 \end{bmatrix} \tag{7.1}$$

(i.e. by the columns of the lower half of the matrix N). The proof is analogous to the one given in §4 for the twelve primes of $\bar{W}$. Multiplying these two matrices on the left by $\Pi^{-1} = \begin{pmatrix} Q & j \\ j^T & 0 \end{pmatrix}$ gives the transpose of the matrix (5.9) and the matrix I_{12}. Hence, under the *polarity* on $\mathscr{C}$ with matrix

$$\Pi = \begin{pmatrix} Z & j \\ j^T & 1 \end{pmatrix}, \tag{7.2}$$

the polars of the 24 points $N_i(i \in \Omega)$ are the 24 primes $\pi_i(i \in \Omega)$ (in order). (It is misleading in the context of a projective space over $GF(2)$ to speak of a 'quadric'). The configuration of 24 points and 24 primes in $PG(11,2)$ is *self-dual*.

The group M_{24} of collineations on $\bar{V}$, which permutes the vertices of the reference simplex, is given by (5.10). Of course, it induces the group M_{24} in $\mathscr{C}$ that permutes the 24 primes:

$$A: \pi_i \to \pi_{i\alpha}, \quad B: \pi_i \to \pi_{i\beta}, \quad H: \pi_i \to \pi_{i\gamma\delta}. \tag{7.3}$$

The M_{24} generated by A, B and H permutes the 24 primes, and leaves the point N_∞ fixed, the M_{24} generated by A, B and F permutes the 24 points and leaves π_∞ fixed, and the polarity Π interchanges the points and primes.

The two (reducible) twelve-dimensional representations of M_{24}, with α, β, $\gamma\delta$ represented, respectively, by A, B, H and by A^{-1}, B, F are adjoint to each other. That is,

$$\Pi^{-1}A\Pi = A^T, \quad \Pi^{-1}B\Pi = (B^{-1})^T, \quad \Pi^{-1}H\Pi = (F^{-1})^T. \tag{7.4}$$

Note added in proof

For a comprehensive treatment of coding theory, including properties of the Mathieu groups and other allied topics, see Conway and Sloane [18].

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