GAUGE THEORIES OF GRAVITY

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A review of the historical background to the development of gauge theories of gravity, beginning with the gauge transformation of Maxwell's theory and the early attempt of Weyl to unify electromagenetism and gravitation. The Yang-Mills approach to gauge theories of internal symmetries is discussed, indicating how this led to the electro-weak unification, and to attempts at grand unification. The gauging of the group of tetrad rotations, and Kibble's gauging of the Poincaré group are then introduced. Hehl's Poincaré gauge theory is described and we conclude by mentioning some of the possible generalisations and extensions of Poincaré gauge theories.

In the early part of this century, after the advent of Einstein's theory of General Relativity (GR) in 1916,there were just two fundamental forces known and understood by physicists. One was gravity, described by Einstein's new theory. The other was electromagnetism, described by Maxwell's theory. There followed several attempts to unify these two forces - to formulate a more profound theory in which gravity and electromagnetism would be seen to be two aspects of a single phenomenon. Three notable attempts were Weyl's theory[1], Kaluza-Klein theory[2], and Einstein-Schrödinger theory[3]. None of these attempts was successful, but each has some intriguing aspects that remain important in the more recent search for unified theories.

Weyl's theory of 1918 is a generalisation of GR, constructed so as to be invariant under spacetime-dependent changes of scale; i.e., under the transformation $g_{ij} \rightarrow \lambda^2 g_{ij}$ applied to the spacetime metric, with spacetime-dependent scale parameter λ. [Incidentally, Weyl called this invariance (in German) *Eichinvarianz*, which translates to the English 'gauge invariance'. This is the origin of our present terminology for invariance laws with spacetime dependent parameters]. The Christoffel symbols are not invariant under Weyl's transformation, but if you introduce an axiliary vector field ϕ_i which transforms according to

$$\phi_i \rightarrow \phi_i + \partial_i\alpha \ , \ \alpha = \ln\lambda \ ,$$

you can construct an invariant connection

$$\{{}^{k}_{ij}\} + g_{ij}\phi^k - \partial_i{}^k\phi_j - \partial_j{}^k\phi_i \ .$$

Weyl's idea was to identify ϕ_i with the electromagnetic potential. It soon became apparent, however, that Weyl's theory could not be interpreted as a theory of electromagnetism, and the theory was abandoned.

The Kaluza-Klein theory generalises GR by increasing the number of dimensions of spacetime to five, and identifying the extra components of the metric as the electromagnetic potential. This idea does work out successfully, but does not provide a *unified* theory in the true sense. Generalisations of the Kaluza-Klein theory are still actively being investigated as part of the present search for unification.

The Einstein-Schrödinger theory generalises GR by dropping the requirement that g_{ij} shall be symmetric, the idea being that the additional 6 components might describe the electromagnetic fields. This approach has now been completely abandoned. However, the *connection* $\Gamma_{ij}{}^k$ in the Einstein-Schrödinger theory was also asymmetric, and this still remains a viable possibility, as we shall see.

In the early attempts at unification, the strong forces that bind nuclei, and the weak forces governing decay processes, were unknown. The present attempts at unification have to be more ambitious - one would like to see gravitational, electromagnetic, weak and strong interactions all emerge from a single theory.

We can illustrate what is meant by a 'gauge theory' by looking into the simplest gauge theory, Maxwell's theory of electromagnetism. Electromagnetism is described by a four-vector *potential* A, and the electormagnetic *fields* F are described by its curl:

$$F_{ij} = \partial_i A_j - \partial_j A_i \ . \qquad (1)$$

Since the potential A occurs in Maxwell's equations only in the combination F, the equations are invariant under the *gauge transformation*

$$A_i \rightarrow A_i + \partial_i \alpha \ , \qquad (2)$$

Where α is an arbitrary spacetime-dependent parameter. A deep insight into the significance of this is provided by quantum mechanics (1925). Observables are constructed from a wave function ψ and its conjugate $\psi^{\dagger}$ and are invariant under the phase transformation

$$\psi \rightarrow e^{i\alpha}\psi \ . \qquad (3)$$

One can then argue that, since there is no *a priori* way of comparing the phases of ψ at two different points,physical laws should still be invariant under such a transformation even if the parameter α is spacetime-dependent. This more general invariance can be achieved by introducing an auxiliary field A_i, transforming according to (2), when ψ undergoes the transformation (3),and replacing the derivatives $\partial_i\psi$ in the wave equation of ψ by the generalised derivatives

$$\partial_i \psi - iA_i\psi \ . \qquad (4)$$

The auxiliary field A is identified as the electromagnetic field, and the modification (4) leads to minimal coupling between a charged field and the photon field A.

According to Noether's theorem (1910), invariance of a Lagrangian theory under a transformation implies a *conservation law*. For the phase transformation (3), the acompanying conservation law is the conservation of electric charge, which can be written as a continuity equation,

$$\partial_i J^i = 0 , \quad (5)$$

satisfied by the charge current. When the phase transformation is *gauged* (i.e., made to depend on position in space and time), the current becomes the *source* of the electromagnetic field:

$$\partial_i F^{ij} = J^j . \quad (6)$$

This is an important feature of all gauge theories. When a symmetry of a Lagrangian theory is gauged, the conserved Noether currents associated with the symmetry become the sources of the axuiliary fields (gauge potentials).

The gauge group of Maxwell's theory is the one-parameter Abelian group U (1). Yang and Mills[4](1954) applied the gauge principle to a non-Abelian group, namely the 3-parameter group SU(2) of isotopic spin. Isotopic spin (nowadays called simply 'isospin') was invented by Heisenberg[5] in 1932, to provide a theoretical understanding of the observed *charge independence of nuclear forces*. One observes that proton and neutron have nearly the same mass,and that the forces that bind protons and neutrons in a nucleus do not appear to recognize any distinction between them; the strong interaction behaves as if proton and neutron are two states of the same particle - the nucleon. A nucleon wave function can be written

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix} . \quad (7)$$

If the electromagnetic interaction (and the weak interaction) could be 'switched off', then any linear combination of the two nucleon states could be chosen and called a 'proton state'; the orthogonal state would be a 'neutron state'. This would be purely a matter of convention in a world with no means of making a distinction. Thus, physical laws appertaining to the strong interaction should be invariant under

$$\psi \rightarrow S\psi \tag{8}$$

where S is any unitary unimodular 2×2 matrix. We have a symmetry group SU(2). The associated Noether conservation law is the *conservation of isotopic spin* (which, of course, is violated by the electromagnetic interaction. It is also violated by the weak interaction). Yang and Mills argued that, in the world of the strong interaction, the choice of convention whereby one of the indistinguishable states of the nucleon is called 'proton', can be made independently at different points. Or, in other words, the strong interactions should remain invariant under (8) *even when S is spacetime-dependent.* To achieve this, they introduce an auxiliary vector field B with transformation

$$B_i \rightarrow SB_iS^{-1} + (\partial_iS)S^{-1} \tag{9}$$

and replace derivatives $\partial_i\psi$ of the nucleon wave functions by the generalised derivatives

$$D_i\psi = \partial_i\psi - B_i\psi \ . \tag{10}$$

Each component of B_i is a traceless Hermitian 2×2 matrix:

$$B_i = \underset{\sim}{b}_i \cdot \underset{\sim}{T} \tag{11}$$

where $\underset{\sim}{T}$ represents Heinsenberg's three isospin matrices . Equation (10) gives the generalised derivative of an isospin 1/2 field. Other fields that interact strongly belong to different representations of the isospin SU(2); the pion field for instance is an isospin-1 field. In every case a generalised derivative is readily constructed. The B-field is the *gauge potential* for the isospin group SU(2). The *gauge fields*

$$F_{ij} = \underset{\sim}{F}_{ij} \cdot \underset{\sim}{T}$$

are defined by

$$[D_i, D_j] = -F_{ij} \tag{12}$$

and are found to be given by a generalised curl

$$F_{ij} = \partial_i B_j - \partial_j B_i - [B_i, B_j]. \qquad (13)$$

They transform homogeneously,

$$F_{ij} \rightarrow S F_{ij} S^{-1} \qquad (14)$$

and satisfy an identity

$$D_i F_{jk} + D_j F_{ki} + D_k F_{ij} = 0 \qquad (15)$$

(as a consequence of the Jacobi identity $[D_i[D_j D_k]] + [D_j[D_k D_i]] + [D_k[D_i D_j]] = 0$). The Lagrangian for the gauge potential B was chosen by Yang and Mills, by analogy with Maxwell's theory, to be

$$-1/8 \text{ trace } F_{ij}F^{ij} = -\tfrac{1}{4} \underset{\rightarrow}{F}_{ij} \cdot \underset{\rightarrow}{F}^{ij}$$

and the field equations are then

$$D_i F^{ij} = I^j \quad , \qquad (16)$$

where $I^i = \underset{\rightarrow}{I}^i \cdot \underset{\rightarrow}{T}$ is the *isospin current*. It satisfies

$$D_i I^i = 0 \qquad (17)$$

instead of a true continuity equation. This is because I^i is the current that carries the isospin of all fields *other than* B. The isospin of B itself has been left out. To get a true continuity equation and hence a conservation law for isospin, the isospin current of B has to be included. This can be done as follows. Equation (16) is

$$\partial_i F^{ij} = I^j + [B_i, F^{ij}]$$

which leads to

$$\partial_j (I^j + [B_i, F^{ij}]) = 0$$

The term $[B_i, F^{ij}]$ is the isospin current of B. This extra complication is characteristic of a gauge theory of a *non-Abelian* group. In the Abelian gauge theory of electromagnetism, photons couple to all particles that carry electric charge, but do not themselves carry electric charge; they are not self-interacting and so their field equations are linear. In the non-Abelian theory of Yang and Mills, the B-quanta couple to all particles that carry isospin. The B-quanta themselves carry isospin (B is an isospin-1 field). They are therefore self interacting and so their field equations are *nonlinear*. The nonlinearity enters the equations through the final term in (13).

The Yang-Mills theory implied the existance of B-quanta, mesons with spin-1 and isospin-1 (three charge states). No such particles were known when the theory was proposed. The ρ-mesons which have spin-1 and isospin-1, might be identified with the Yang-Mills B quanta. However, the importance of the Yang-Mills paper does not reside in this prediction. The work of Yang and Mills is of historic importance because it revealed a general principle - the concept of *gauging a non-Abelian symmetry group*. Applications of this principle have been remarkably successful. Indeed, in recent years, non-Abelian gauge theories have become a dominant theme in the search for an understanding of the nature of fundamental physical forces.

The Salam-Weinberg theory[6] of 1967 successfully unified the weak and electromagnetic interactions. It is a gauge theory based on a group SU(2) ('weak isospin') ⊗ U(1)('weak hypercharge'), which employs the principle of *spontaneous symmetry breaking* (SSB). In a traditional Lagrangian field theory, all the fields are zero in the 'vacuum state' or 'ground state' (minimum energy state). But Lagrangian theories are possible in which there are fields that have nonvanishing components, in the vacuum state. These special fields are the Higgs fields. When this happens,the symmetry of the physics is less than the symmetry of the Lagrangian, since the symmetry transformations have to respect the structure of the peculiar vacuum. This phenomenon is spontaneous symmetry breaking. There is a theorem, due to Goldstone[7], that whenever a symmetry is spontaneously broken in this way, some spinless massless fields will be present. This seemed to rule out SSB in nature, because nature does not produce any spinless massless fundamental particles. This difficulty was eliminated by Higg's discovery[8] that, if the symmetry that gets spontaneously broken is a *gauge* symmetry, spinless massless 'Goldstone bosons' would not be present. *Instead*, the particles would acquire mass (they cannot have a mass term in the Lagrangian without spoiling the gauge invariance). Thus the 'Higgs mechanism' kills two birds with one stone: it eliminates unwanted Goldstone fields from theories with SSB, and it explains how gauge quanta can have a mass. In the Salam-Weinberg theory, the $SU(2) \otimes U(1)$ gauge symmetry of the Lagrangian is spontaneously broken,so that the exact gauge symmetry of the physics is only a $U(1)$, with massless gauge potential A. This is electromagnetism. The other gauge potentials $W^{\pm}$ and Z become massive through the Higgs mechanism. They are the intermediate bosons of the weak interaction. The charged W-bosons mediate the V-A theory of Marshak and Sudarshan[9]. The Salam-Weinberg theory predicts an approximate mass for them, and they have now been detected experimentally. The neutral Z-bosons couple to neutral weak currents, and experimental evidence indicates that this aspect of the weak interactions appears to behave as expected according to the theory. Thus, the gauge theory of Salam and Weinberg is a remarkably successful theory.

Some aspects of the strong interactions appear to be well described by quantam chromodynamics (QCD), which is also a gauge theory. According to QCD, three quark fields (red, yellow and blue) are coupled to 'gluons', the massless gauge quanta of a symmetry group $SU(3)_{colour}$.

The so-called grand unified theories[10] (GUT's) are attempts to unify these two successful theories, the Salam-Weinberg theory and the QCD approach to strong interactions, thereby unifying the electromagnetic, the weak and the strong interactions. The idea is the following: Incorporate the Salam-Weinberg $SU(2) \otimes U(1)$ and the $SU(3)_{colour}$ of QCD in a bigger group G (which should be semi-simple so that the coupling of its gauge potentials to other fields needs only one coupling constant). Then make a gauge theory of G, incorporating various SSB mechanisms. Then hope that the resulting theory will correctly describe experimental data.

Of course, the GUT approach to unification is not ambitious enough, if the aim is to unify *all* the fundamental forces of nature. It leaves out the gravitational interaction completely. It now appears extremely likely that the long-sought unified theory of all interactions (if such is possible) will be a gauge theory. So a necessary ingredient is a gauge theory of gravity.

Is Einstein's gravitational theory (GR) a gauge theory? It certainly has some intriguing resemblances to the theory of Yang and Mills. The role of the connection $\Gamma_{ij}{}^{k}$ in the construction of covariant derivatives is analogous to the role of B_i in constructing the generalised derivative (10) and the inhomogeneous transformation law of the connection under general coordinate transformations (GCT's) corresponds to the inhomogeneous transformation law (9). Consider also the resemblance between (13) and the structure of the curvature tensor:

$$R_{ijk}{}^{l} = \partial_i\Gamma_{jk}{}^{l} - \partial_j\Gamma_{ik}{}^{l} - \Gamma_{ik}{}^{m}\Gamma_{jm}{}^{l} + \Gamma_{jk}{}^{m}\Gamma_{im}{}^{l} \qquad (18)$$

The analogue of the identity (15), in GR, is of course the *Bianchi identity*. In fact, if the traceless Hermitian 2×2 matrices B_i are replaced by real 4×4 matrices, the resemblances mentioned above are complete. So GR looks something like a gauge theory of GL(4,R). In this interpretation, the spacetime-dependent GL(4,R) matrices would be the

$$\frac{\partial x'^{i}}{\partial x^{j}} \qquad (19)$$

that appears in the transformation laws of vectors and tensors under a GCT. *However,* a gauged GL(4,R) should have 16 spacetime-dependent parameters. Because of

$$\frac{\partial^2 x'^i}{\partial x^j \partial x^k} = \frac{\partial^2 x'^i}{\partial x^k \partial x^j} , \qquad (20)$$

the GCT group has *only four*. This is most easily seen by looking at an infinitesimal GCT,

$$x^i \rightarrow x^i - \xi^i ,$$

which has just four spacetime-dependent parameters ξ^i . So GR is *not* a gauge theory of GL(4,R). Moreover, in a true gauge theory, the gauge potential should be a genuine dynamical field in its own right. The connection in GR is not; it is constructed from the metric:

$$\Gamma_{ij}^{\ \ k} = \{ {}^{k}_{ij} \} \qquad (21)$$

So we conclude that GR is not a gauge theory of GL(4,R), though it resembles one to some extent. The question that then arises is: is it possible to construct a true gauge theory that will describe gravitational interactions, and that will pass the celebrated 'three tests'?

It is possible to formulate GR in terms of a *tetrad* - a set of four orthonormal fields $e_\alpha^{\ i}$ (here, the Latin index is the coordinate based index and the Greek index labels the four vectors). The inverse of the matrix of components will be written $e_i^{\ \alpha}$. Orthonormality means that

$$e_\alpha^{\ i} e_\beta^{\ j} g_{ij} = \eta_{\alpha\beta} \qquad (22)$$

where $\eta_{\alpha\beta}$ is the Minkowski metric of special relativity. Now, if you change the tetrad,

$$e_\alpha^{\ i} \rightarrow \Lambda_\alpha^{\ \beta} e_\beta^{\ i} , \qquad (23)$$

the matrix Λ will be a Lorentz matrix, so as to preserve the orthonormality. It can be a *spacetime-dependent* matrix, so we have the beginings of a gauge theory for the Lorentz group.

If all the physical fields are vectors and tensors, the tetrad approach is only a reformulation of GR and says nothing new. However, to investigate the behaviour of fermions in an Einstein gravitational field, we have to introduce spinor fields, and then the tetrad formulation of GR become indispensible. Under a change of tetrad (23), a spinor field ψ transforms according to

$$\psi \quad \to \quad S\psi \tag{24}$$

where S is determined (up to a sign) by $|S| = 1$ and

$$S^{-1}\gamma_\alpha S \quad = \quad \Lambda_\alpha{}^\beta\gamma_\beta \tag{25}$$

We have a matrix group SL(2,C) with spacetime-dependent parameters. Invariance of the Dirac equation is maintained by replacing $\partial_i\psi$ by

$$D_i\psi \quad = \quad \partial_i\psi \; - \; \Gamma_i\psi \tag{26}$$

where the *Fock-Ivanenko*[11] *coefficients* Γ_i transform according to

$$\Gamma_i \quad \to \quad S\Gamma_i S^{-1} \; + \; (\partial_i S)S^{-1}. \tag{27}$$

They are 4×4 matrices of the form

$$\Gamma_i \;=\; \tfrac{1}{4}\Gamma_i{}^{\alpha\beta}\sigma_{\alpha\beta} \quad , \quad \sigma_{\alpha\beta} = \tfrac{1}{2}[\gamma_\alpha,\gamma_\beta] \tag{28}$$

(c.f. (11)). The components $\Gamma_i{}^{\alpha\beta}$ are *spin coefficients*. All this was written down by Fock and Ivanenko in 1929, and it looks very much like a gauge theory of SL(2,C). However, these authors were not trying to construct a new theory or to generalise GR, they wished only to understand how Dirac's (then new) theory of the electron could be reconciled with the curved spacetime of Einsteinian gravity, so that the behaviour of an electron in a gravitational field could be understood. If you have a tetrad $e_i{}^\alpha$ and a set of spin coefficients $\Gamma_i{}^{\alpha\beta}$ you can build a connection $\Gamma_{ij}{}^k$:

$$\partial_i e_j{}^\alpha \; + \; e_j{}^\beta\Gamma_{i\beta}{}^\alpha \quad = \quad \Gamma_{ij}{}^k e_k{}^\alpha \; . \tag{29}$$

At this point, contact with GR is made by insisting that

$$\Gamma_{ij}{}^{k} = \{{}^{k}_{ij}\} . \tag{30}$$

The Fock-Ivanenko coefficients are thus not independent dynamical fields. They are built out of the tetrad and its derivatives.

Schrödinger[12] (1932) wrote

$$\Phi_{ij} = \partial_i\Gamma_j - \partial_j\Gamma_i - [\Gamma_i,\Gamma_j] \tag{31}$$

(c.f.(13)!) and pointed out that, on account of (30),

$$\Phi_{ij} = \tfrac{1}{4}R_{ij}{}^{\alpha\beta}\sigma_{\alpha\beta} \tag{32}$$

where the coefficients are the components of the Riemann tensor constructed from the Christoffel symbols. In the same paper, he also pointed out that you could incorporate electromagnetism by adding iA_i to the definition of Γ_i ; then iF_{ij} gets added to (32).

With hindsight, it is possible to see that the constraint (30) was a blunder. Utiyama (1956) and Sciama (1962) presented a true gauge theory of SL(2,C)[13]. Their approach resembles the earlier work of Fock and Ivanenko and Schrodinger, but the constraint (30) is abandoned. Instead, the spin coefficients $\Gamma_i{}^{\alpha\beta}$ are components of a true dynamical field (SL(2,C) gauge potentials). These and the tetrad components $e_i{}^{\alpha}$ are the quantities to be varied in the Lagrangian to get the Euler-Lagrange equations. The connection defined through (29) satisfies

$$\partial_k g_{ij} - \Gamma_{ki}{}^{l} g_{lj} - \Gamma_{kj}{}^{l} g_{il} = 0 , \tag{33}$$

as in Einstein's theory (the connection is 'metric compatible'), but it is *not* (in general) *symmetric*. We have a spacetime with *torsion*.

$$T_{ij}{}^{k} = \Gamma_{ij}{}^{k} - \Gamma_{ji}{}^{k} . \tag{34}$$

The theory can be reformulated in terms of g_{ij} and $T_{ij}{}^{k}$ and is then seen to be the same as the earlier *Einstein-Cartan* theory[14].

Let us at this stage reiterate the Yang-Mills recipe for producing a gauge theory. Take a symmetry group of a Lagrangian field theory and require that the invariance shall still hold even when the parameters of the symmetry group are made spacetime-dependent. To achieve this, introduce auxiliary fields (gauge potentials) and modify derivatives of all other fields that respond to the symmetry transformations. Then construct an invariant Lagrangian for the gauge potentials. Standard Yang-Mills-type theories are based on *internal* symmetry groups, like isospin SU(2), which act on components of fields but have to effect on the points of spacetime. Now, there is symmetry group of fundamental importance that does act on the points of spacetime: the 10-parameter Poincaré group of Lorentz rotations and translations in the Minkowski space of special relativity. Kibble (1971)applied the Yang-Mills prescription to this *spacetime* symmetry[15]. The theory that he obtained is identical to that given by Sciama; it is now known as the Kibble-Sciama theory, or Einstein-Cartan-Kibble-Sciama theory (ECKS).

The action of the Poincaré group on Minkowski space and on the components of a field ψ can be written

$$x'^{\alpha} = x^{\beta}\Lambda^{-1}{}_{\beta}{}^{\alpha} - a^{\alpha} \tag{35}$$

$$\psi'(x') = S(\Lambda)\psi(x), \tag{36}$$

where S is a representation of the Lorentz subgroup. The Noether currents for this symmetry are the canonical energy-momentum current $\theta_{\alpha}{}^{i}$ and the canonical angular momentum current

$$x_{\alpha}\theta_{\beta}{}^{i} - x_{\beta}\theta_{\alpha}{}^{i} + \tau_{\alpha\beta}{}^{i}$$

($\tau_{\alpha\beta}{}^{i}$ is the current that carries *intrinsic* angular momentum, or spin). Conservation of energy,momentum and angular momentum are

$$\partial_i \theta_\alpha{}^i = 0,$$

$$\partial_i \tau_{\alpha\beta}{}^i = \theta_{\alpha\beta} - \theta_{\beta\alpha} \tag{37}$$

In a gauge theory of the Poincaré group, these are the currents that one expects to see as *sources* of the gauge potentials. The canonical energymomentum tensor is not in general symmetric. The symmetrised energy-momentum tensor of Belinfante is

$$T^{ij} = \theta^{ij} - \tfrac{1}{2}\partial_k(\tau^{ijk} + \tau^{kij} + \tau^{kji}) \tag{38}$$

(its symmetry $T^{ij} = T^{ji}$ and conservation law $\partial_i T^{ij} = 0$ both follow from the Noether identities (37)). The energy momentum tensor of GR that acts as a source for the gravitational field (the right-hand side of Einstein's equations) is just the straightforward covariant generalisation of Belinfante's tensor[16].

When the Poincaré group is gauged by making the parameters a^α and $\Lambda_\alpha{}^\beta$ spacetime dependent, (35) becomes just a *general coordinate transformation* (36) becomes a combination of this GCT with a *change of tetrad* (23). The gauge potentials of the Poincaré gauge theory consist of a tetrad $e_i{}^\alpha$ (gauge potentials for the translational subgroup) and a set of spin coefficients $\Gamma_i{}^{\alpha\beta}$ (gauge potentials for the Lorentz subgroup). So, in Kibble's approach, the tetrad appears as an integral part of the Yang-Mills principle.

The gauge-potential of the Poincaré group can be combined in a single quantity

$$\Gamma_i = e_i{}^\alpha G_\alpha + \tfrac{1}{2}\Gamma_i{}^{\alpha\beta} G_{\alpha\beta} \tag{39}$$

where G_α and $G_{\alpha\beta} = -G_{\beta\alpha}$ are generators of the Lie algebra of the Poincaré group (c.f.(11)). They satisfy

$$[G_\alpha, G_\beta] = 0$$

$$[G_{\alpha\beta}, G_{\gamma\delta}] = \eta_{\beta\gamma}G_{\alpha\delta} - \eta_{\alpha\gamma}G_{\beta\delta} + \eta_{\alpha\delta}G_{\beta\gamma} - \eta_{\beta\delta}G_{\alpha\gamma}$$

$$[G_{\alpha}, G_{\beta\gamma}] = \eta_{\alpha\beta}G_{\gamma} - \eta_{\alpha\gamma}G_{\beta} \tag{40}$$

One can then work out the components of the gauge fields

$$F_{ij} = F_{ij}{}^{\alpha}G_{\alpha} + \tfrac{1}{2}F_{ij}{}^{\alpha\beta}G_{\alpha\beta} = \partial_i\Gamma_j - \partial_j\Gamma_i - [\Gamma_i, \Gamma_j]. \tag{41}$$

One finds that the *translational gauge field* is just the *torsion* of the connection $\Gamma_{ij}{}^{k}$ given by (29), and that the *rotational gauge field* is the *curvature* of this connection:

$$F_{ij}{}^{k} = \Gamma_{ij}{}^{k} - \Gamma_{ji}{}^{k}$$

$$F_{ijk}{}^{l} = \partial_i\Gamma_{jk}{}^{l} - \partial_j\Gamma_{ik}{}^{l} - \Gamma_{ik}{}^{m}\Gamma_{jm}{}^{l} + \Gamma_{jk}{}^{m}\Gamma_{im}{}^{l} . \tag{42}$$

The Lagrangian for the gauge potentials was chosen by both Kibble and Sciama to be the straightforward generalisation of Einstein's Lagrangian: they chose

$$\frac{1}{2\kappa} \, e e_{\alpha}{}^{j} e_{\beta}{}^{i} F_{ij}{}^{\alpha\beta} \tag{43}$$

where

$$e = |e_i{}^{\alpha}| = \sqrt{-g} .$$

Varying $e_i{}^{\alpha}$ and $\Gamma_i{}^{\alpha\beta}$, they obtained field equations

$$F_{ij} - \tfrac{1}{2} g_{ij} F = \kappa\theta_{ij} , \tag{44}$$

$$F_{ijk} + F_i g_{jk} - F_j g_{ik} = \kappa\tau_{ijk} \tag{45}$$

($F_{ij} = F_{kij}{}^{k}$, $F = F_i{}^{i}$, $F_i = F_{ij}{}^{j}$). *Observe that the sources are the Noether currents,* as expected. Equations (44) look like Einstein's equations, but here the Ricci tensor and the energy-momentum tensor are not in general symmetric. Equation (45) is *not* a differential equation, so in this theory the torsion cannot propagate through space it is nonzero only in the presence of matter with spin. In fact, if there is no intrinsic spin, the torsion vanishes. In conjunction with (33) this implies that the connection is, in that case, just the Christoffel connection. Moreover, if there is no intrinsic spin the canonical energy momentum tensor is the symmetric Bélinfante tensor. Thus the theory is identical to Einstein's theory except in regions where there is matter with intrinsic spin. Moreover, one can through (45) express torsion in terms of the spin current, and substitute spin for torsion in the Lagrangian. It then turns out that the theory is still identical to Einstein's theory but with extra small terms representing spin-spin coupling.

Hehl[17] has proposed that the Lagrangian chosen by Sciama and Kibble is not the correct one, and that the Lagrangian of the true Poincare gauge theory should be a quadratic formed from the gauge fields (analogous to the $\frac{1}{4}F_{ij}F^{ij}$ of Maxwell'stheory). Poincare gauge theories of this kind have been investigated extensively by Hehl and his co-workers in recent years. The curvature-squared terms are believed to describe some aspects of the strong interactions, the classical gravitational interaction coming from the torsion-squared part of the Lagrangian, the most general form of which is

$$e(C_1F_{ijk}F^{ijk} + C_2F_{ijk}F^{ikj} + C_3F_iF^i). \tag{46}$$

Let us consider just those solutions for which $\Gamma_i{}^{\alpha\beta} = 0$. Then $F_{ij}{}^{\alpha\beta} = 0$ and

$$F_{ij}{}^{\alpha} = \partial_i e_j{}^{\alpha} - \partial_j e_i{}^{\alpha} . \tag{47}$$

We have a space with vanishing curvature but nonvanishing torsion. Such spaces were called by Einstein 'spaces of distant parallelism'. The 3-parameter family of Lagrangian densities (46) provides theories that can be regarded as *gauge theories of the translational subgroup* of the Poincare group (a 4-parameter Abelian group). *One of these is Einstein's* *theory;* from (22) and (47) one can obtain an identity

$$\sqrt{-g}R = e(\tfrac{1}{4}F_{ijk}F^{ijk} - \tfrac{1}{2}F_{ijk}F^{ikj} - F_iF^i) + D \qquad (48)$$

where D is a divergence and so does not affect the field equations. In fact this is the combination that you get uniquely if you insist that the translational gauge Lagrangian be invariant under tetrad changes (23)[18]. However, other choices of the parameters in (46) also lead to viable gravitational theories, in that the resulting theory possesses a Schwarzschild solution and is not distinguishable, observationally, from Einstein's theory. The Poincaré gauge theory favoured by Hehl, and investigated extensively by him and his co-workers, is based on the combination

$$e(F_{ijk}F^{ijk} - 2F_iF^i) \; . \qquad (49)$$

Variations on the theme that we have presented are numerous. Instead of the action of the Poincare group on Minkowski space, one can consider the action of the de Sitter group on de Sitter space, and gauge that [20,21]. One can extend the Poincare action on Minkowski space to the action of the 16-parameter affine group, and gauge that[22]. Or one might construct a gauge theory of the 15-parameter conformal group[23] (transformations on Minkowski space that preserve light cones). Since these latter two groups are not invariance groups for physical laws, their gauge theories should be supplemented by SSB. Groups that unite spacetime symmetry and internal symmetry in a non-trivial way can be explored[24]. Gauge theories of internal symmetries have spin-1 gauge quanta. Gauge theories of spacetime symmetries also have spin-2 gauge quanta (gravitons) and incorporate gravitational effects.

In the search for the ultimate unified theory that would incorporate all the fundamental forces of nature, the contribution to be made by investigating gauge theories of spacetime symmetries appears indispensible.

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