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A Unified Approach to the Gauging of Space-Time and Internal Symmetries

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The properties of the manifold of a Lie group G, fibered by the cosets of a subgroup H, are exploited to obtain a geometrical description of gauge theories in space-time G/H. Gauge potentials and matter fields are pullbacks of equivariant fields on G. Our concept of a connection is more restricted than that in the similar scheme of Ne'eman and Regge, so that its degrees of freedom are just those of a set of gauge potentials for G, on G/H, with no redundant components. The "translational" gauge potentials give rise in a natural way to a nonsingular tetrad on G/H. The underlying group G to be gauged is the group G of left translations on the manifold G and is associated with a "trivial" connection, namely the Maurer–Cartan form. Gauge transformations are all those diffeomorphisms on G that preserve the fiber-bundle structure.

1. INTRODUCTION

The language of fiber bundles provides very powerful geometrical methods of dealing with Yang–Mills theories (gauge theories). We refer the reader to the review articles of Daniel and Viallet [1] and of Eguchi, Gilkey, and Hanson [2]. In the "conventional" approach to the fiber-bundle description of the gauge theory of an internal symmetry group G, The Yang–Mills potential on space-time M is interpreted as the pullback (from a section) of a connection 1-form on the principal fiber bundle $P(M, G)$. Gauge transformations are interpreted as changes of section. This conventional approach leads to complications and conceptual difficulties when extended to describe a gauge theory of a space-time symmetry group [3, 4]. It requires the introduction of additional structures, such as affine frames [4, 5] in the

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case of Poincaré and affine gauge theories and second-order frames in the case of conformal gauge theories [6]. A more serious flaw is that the identification of the translational gauge potential with a tetrad on M, which is perfectly natural and desirable in, for example, Poincaré gauge theory, loses its validity in the conventional fiber-bundle description [3, 4]. This indicates that the conventional approach to the fiber bundle description of a gauge theory is inappropriate if the underlying group acts on space-time. An alternative approach is accordingly developed here that does not encounter these difficulties and that provides an elegant unified description of gauge theories of groups involving space-time symmetries as well as internal symmetries.

Our scheme is based on the geometrical properties of a principal fiber bundle $G(G/H, H)$, with H a subgroup of G and G/H interpreted as space-time. The idea of describing gauge theories of space-time symmetries in terms of this kind of bundle has been previously explored by Ne'eman and Regge [7]; this scheme differs from theirs in the concepts of gauge transformation and connection. Poincaré gauge theories are readily generalizable to gauge theories of groups involving other space-time symmetries as well as internal symmetries, as we have shown elsewhere [8]. The present work is the appropriate fiber-bundle description of this general formulation of gauge theories. The groups G and H are Lie groups, but otherwise no requirements are imposed on them. They do not have to be compact, semisimple, etc. In particular, and of special importance, they are not necessarily connected Lie groups. Thus the important discrete space-time symmetries parity, time reversal and (in the case of the conformal group) inversion, have a natural place in this scheme.

A more detailed discussion, making use of appropriate coordinate systems on $G(G/H, H)$ and the corresponding component sets of the geometrical fields, will be published elsewhere. In the present work (apart from one lapse in Sec. 3), a coordinate-free notation is used throughout.

We mean, by G, a differentiable manifold together with a differentiable mapping $G \times G \to G$ satisfying the group axioms, and we mean by G/H the set of left cosets, which inherits a differentiable manifold structure from G. No further structure is implied in our definition of the principle fiber bundle $G(G/H, H)$. The structure group of this bundle is H, acting on the right.

To avoid misunderstandings, we wish to make clear at the outset that our construction of the gauge theory of the group G *involves no modification of the underlying bundle space* $G(G/H, H)$. Gauge transformations are defined to be the automorphisms of this bundle; that is, they are the diffeomorphisms that preserve fibers and preserve the action of H on fibers. They constitute an infinite-parameter group. The finite-parameter subgroup that preserves the Maurer–Cartan form is isomorphic to G, and in fact consists of the left actions of G on itself. Contact with gauge theories is made by putting a generalized connection on G—an equivariant vielbein or equivalently an equivariant 1-form valued in the Lie algebra of G. Associated with any generalized connection is a set of canonical infinitesimal generators Q_A of gauge transformations, satisfying a distorted form of the commutation relations of the Lie algebra of G.

2. GEOMETRY OF A LIE GROUP

In this section some basic concepts are reviewed in order to provide a background and establish the notation for our scheme.

Denote the general element of a Lie group G by z and a particular element by g. The left translation associated with g (or the left action of g on G), the right translation associated with g, the adjoint action of g, and the inversion mapping are the diffeomorphisms on the group manifold defined by

$$L_g z = gz, \qquad R_g z = zg, \qquad D_g z = gzg^{-1}, \qquad Nz = z^{-1} \tag{1}$$

Left translations commute with right translations. Moreover, if f is a diffeomorphism that commutes with all right translations

$$f(zg) = f(z)\, g$$

for every $g \in G$, then f is a left translation ($g_0(z) = f(z)\, z^{-1}$ is independent of z, and $f = L_{g_0}$). Therefore, *a diffeomorphism commutes with every right translation if and only if it is a left translation* (and vice-versa).

A vector field generates a one-parameter group of diffeomorphisms. Let X be a right-invariant vector field ($dR_g X = X$ for every $g \in G$) and Y a generator of right translations. Then the infinitesimal transformation law of X implies $[X, Y] = 0$. The diffeomorphisms generated by X therefore commute with all the right translations associated with elements of G_0, the connected subgroup of G. Therefore, X generates a one-parameter group of left translations. *Right-invariant vector fields generate left translations* (and vice-versa).

Since the commutator of two left-invariant vector fields is left-invariant, the left-invariant vector fields constitute an algebra under commutation, the *Lie algebra* $\mathfrak{G}$ of G.

A left-invariant vector field X is uniquely determined by its value at any point (since $dL_g X_z = X_{gz}$). Let X_z be a vector at a point z and let X be the unique left-invariant vector field determined by it. A mapping $\theta_z\colon T_z(G) \to \mathfrak{G}$ can be defined by $\theta_z X_z = X$ (for every vector X_z at z). The collection of these mappings for all $z \in G$ constitute a Lie-algebra valued 1-form θ, the *Maurer–Cartan form.*

A basis for the tangent space at any one point determines a left-invariant vielbein $\{R_A\}$, which we call the *right vielbein.* It provides a basis for the tangent space at every point. The dual bases of the cotangent spaces constitute a set $\{R^A\}$ of left-invariant 1-form fields, satisfying $R^A(R_B) = \delta^A_B$. Since $R_g N = N L_{g^{-1}}$, the vectors

$$L_A = -dNR_A \tag{2}$$

are right-invariant. They constitute the *left vielbein* $\{L_A\}$. The fields R_A provide a basis for $\mathfrak{G}$. The constants $c_{AB}{}^C$ defined by

$$[R_A, R_B] = c_{AB}{}^C R_C \tag{3}$$

are the *structure constants* of G. A matrix representation of $\mathfrak{G}$ is provided by a set of matrices G_A satisfying

$$[G_A, G_B] = c_{AB}{}^C G_C \tag{4}$$

A *matrix representation* of G is a matrix-valued scalar field S satisfying $S(g)\,S(z) = S(gz)$ and $S(z^{-1}) = S^{-1}(z)$, for every $g \in G$, $z \in G$. The behavior of S under the diffeomorphisms (1) is found to be

$$L_g^* S = S(g)\,S, \qquad R_g^* S = SS(g), \qquad D_g^* S = S(g)\,SS(g^{-1}), \qquad N^* S = S^{-1} \tag{5}$$

The *generators* of the representation S (with respect to a particular basis $\{R_A\}$ of $\mathfrak{G}$ are the matrices

$$G_A = S^{-1} R_A S \tag{6}$$

They are left-invariant and therefore *constant* matrices, and they satisfy (4), thus providing a matrix representation of $\mathfrak{G}$. Let S be a matrix representation of G that is faithful for the connected subgroup G_0. There is then no $g \in G_0$, other than the identity element, for which $S(z) = S(z)\,S(g)$ for every $z \in G$. That is, there is no nontrivial right translation associated with an element of G_0 that leaves S invariant. The infinitesimal version of this statement is that there is no nonvanishing left-invariant vector field X such that $X(S) = 0$, or $X^A R_A S = 0$ implies $X^A = 0$. Multiplying by S^{-1} on the left, $X^A G_A = 0$ implies $X^A = 0$. Therefore, *the generators of a matrix representation of G that is faithful for G_0 are linearly independent.*

Given a matrix representation of $\mathfrak{G}$, in which the R_A are represented by matrices G_A, the Maurer-Cartan form θ is represented by a matrix-valued 1-form, or equivalently a matrix of 1-forms, $\theta = G_A \theta^A$ satisfying $\theta_z R_{Az} = G_A$. We use the same symbol, θ, to denote either the Maurer–Cartan form or any matrix representative of it. The coefficients θ^A are ordinary 1-forms. We have $G_B(\theta^B R_A)_z = G_A$. Choosing a representation of $\mathfrak{G}$ with linearly independent generators, this implies $\theta^B R_A = \delta^B_A$, $\theta^A = R^A$, and hence

$$\theta = G_A R^A \tag{7}$$

The matrix-valued 1-form $S^{-1}\,dS$ satisfies $(S^{-1}\,dS)\,R_A = S^{-1} R_A S = G_A$, so

$$\theta = S^{-1}\,dS \tag{8}$$

If G is a matrix group, we can use the self-representation, and write

$$\theta = z^{-1}\,dz \tag{9}$$

The Maurer–Cartan form is left-invariant ($L_g^* \theta = \theta$). Its transformation law under right translations is easily found from (9)

$$R_{g^{-1}}\theta = D_{g^{-1}}\theta = g\theta g^{-1} \tag{10}$$

Writing $z\theta = dz$, we see that $0 = d(z\theta) = dz \wedge \theta + z\,d\theta$. Multiplying by z^{-1} on the left, we get the *Maurer–Cartan equation*

$$d\theta + \theta \wedge \theta = 0 \tag{11}$$

The definition (5) is equivalent to $R_A S = SG_A$. Applying N to this relation and recalling the definition (2), we find $L_A S = G_A S$. From these two relations it is not difficult to deduce that

$$[L_A, L_B] = -c_{AB}{}^C L_C \tag{12}$$

and

$$[R_A, L_B] = 0 \tag{13}$$

Define the scalar fields $D_A{}^B = R^B(L_A)$. Then $D_A{}^B G_B = \theta L_A = S^{-1} L_A S = S^{-1} G_A S$. From this relation

$$D_A{}^B G_B = S^{-1} G_A S \tag{14}$$

one deduces that the matrix field D is a representation of G (the adjoint representation). Substituting (7) into (10), we obtain the transformation law of the 1-form fields R^A under right translations, and hence the transformation law

$$dR_g R_A = dD_g R_A = D_A{}^B(g)\, R_B \tag{15}$$

of the right vielbein.

The generators for the adjoint representation can be found as follows. Employing the Maurer–Cartan equations and the well-known identity $2d\theta(X, Y) = X(\theta Y) - Y(\theta X) - \theta[X, Y]$, satisfied by any 1-form, we find $D_B{}^D c_{DA}{}^C G_C = D_B{}^D[G_D, G_A] = [\theta L_B, \theta R_A] = 2\theta \wedge \theta(L_B, R_A) = 2d\theta(R_A, L_B) = R_A(\theta L_B) = R_A D_B{}^C G_C$ for any representation G_A of the basis of $\mathfrak{G}$. Hence $D_B{}^D c_{DA}{}^C = R_A D_B{}^C$. Defining matrices c_A by

$$(c_A)_D{}^C = c_{DA}{}^C \tag{16}$$

we see that $Dc_A = R_A D$. Comparison with (6) identifies the c_A as the generators of the representation D.

3. LIE-ALGEBRA VALUED FORMS

Let ω be a $\mathfrak{G}$-valued 1-form field on G. Associate with it a derivative operator ∇ that acts on $\mathfrak{G}$-valued p-forms Φ according to

$$\nabla\Phi = d\Phi + \omega \wedge \Phi \tag{17}$$

(The product $\wedge$ here involves the product $\frac{1}{2}[\ ,\ ]$ on $\mathfrak{G}$ as well as the usual wedge product on forms. Explicitly, if $\Phi = \Phi^A G_A$ and $\Psi = \Psi^A G_A$ are two $\mathfrak{G}$-valued forms, then $\Phi \wedge \Psi = \frac{1}{2}\Phi^B \wedge \Psi^C c_{BC}{}^A G_A$). The derivative operator ∇ satisfies

$$\nabla\nabla\Phi = -\tfrac{1}{2}\mathcal{G} \wedge \Phi \tag{18}$$

where

$$\mathcal{G} = -2\nabla\omega \tag{19}$$

This $\mathfrak{G}$-valued 2-form satisfies the *Bianchi identity*

$$\nabla\mathcal{G} = 0 \tag{20}$$

A $\mathfrak{G}$-valued 1-form ω is *nonsingular* if $\omega(X) = 0$ implies $X = 0$. Writing $\omega = E^A G_A$, this means that the 1-forms E^A are linearly independent, and so there exists a dual vielbein $\{E_A\}$ satisfying $E^A(E_B) = \delta^A_B$.

A *local* diffeomorphism with support $U \subset G$ is a diffeomorphic mapping $f\colon U \to fU \subset G$. If ω is a $\mathfrak{G}$-valued 1-form and there exists a local diffeomorphism f with support U such that $\omega = f^*\theta$ on U, then $\mathcal{G} = 0$ on U. That is, on U, $-\frac{1}{2}\mathcal{G} = d\omega + \omega \wedge \omega = d(f^*\theta) + (f^*\theta) \wedge (f^*\theta) = f^*(d\theta + \theta \wedge \theta) = 0$ on account of the Maurer–Cartan equations. Moreover, suppose ω is a nonsingular $\mathfrak{G}$-valued 1-form satisfying $\mathcal{G} = 0$ on some neighborhood of $z_0 \in G$. Then there exist neighborhoods U of z_0 and local diffeomorphisms f with support U such that $\omega = f^*\theta$ on U.

Proof. Put a coordinate chart on a neighborhood of z_0 and denote coordinates of a point z by z^M. Let $z_0{}'$ be any arbitrary point of G and put a chart on a neighborhood of it, denoting coordinates in this chart by z'^M. The existence of a local diffeomorphism satisfying $fz_0 = z_0{}'$, $\omega = f^*\theta$ is equivalent to the existence of functions $f^M(z)$ satisfying $f^M(z_0) = z_0'^M$ and $E^A_N(z) = R^A_M[f(z)]\, \partial f^M/\partial z^N$. This latter differential equation for the f^M can be more concisely written as $R^M_A(f)\, E^A = df^M$. The integrability conditions are $0 = d[R^M_A(f)\, E^A] = R^M_{A\cdot N}(f)\, df^N \wedge E^A + R^M_A(f)\, dE^A$. The first term in this latter expression is $R^M_{A\cdot N}(f)\, R^N_B(f)\, E^B \wedge E^A = \frac{1}{2}[R_B, R_A]^M_f\, E^B \wedge E^A = \frac{1}{2}c_{BC}{}^A R^M_A(f)\, E^B \wedge E^C$. We finally obtain the integrability conditions in the form $0 = R^M_A(f)(dE^A + \frac{1}{2}c_{BC}{}^A E^B \wedge E^C)$. The term in brackets is just the set of components of $\nabla\omega = -\frac{1}{2}\mathcal{G}$. Therefore, if $\mathcal{G} = 0$ in a neighborhood of z_0, there is a neighborhood U of z_0 in which a solution f^M exists. The nonsingularity of ω implies the nonvanishing of the Jacobian $|\partial f^M/\partial z^N|$. The functions f^M therefore specify a local diffeomorphism.

In particular, we have shown that, if ω is a nonsingular $\mathfrak{G}$-valued 1-form, then *ω is locally diffeomorphic to the Maurer–Cartan form in a neighborhood of every point if and only if $\mathcal{G} = -2(d\omega + \omega \wedge \omega) = 0$.*

4. THE FIBER BUNDLE $G(G/H, H)$

Let H be a Lie subgroup of G. The left cosets gH are the orbits of the right action of H on G. They are the fibers of the principal fiber bundle $G(G/H, H)$ with structural group H and base space G/H. A vector tangential to a fiber is called a *vertical* vector; the tangent spaces to the fibers are the vertical spaces. Fibers are mapped to fibers by left translations and hence vertical spaces are mapped to vertical spaces by left translations. Therefore, a vertical vector at any one point of G determines a unique left-invariant vertical vector field. The left-invariant vertical vector fields are the *fundamental fields* on $G(G/H, H)$. They generate the right translations R_h with $h \in H_0$, the connected subgroup of H. They constitute a subalgebra $\mathfrak{v}$ of $\mathfrak{G}$, isomorphic to the Lie algebra of H. A subspace $\mathfrak{a}$ of $\mathfrak{G}$ (not in general a subalgebra), called have the *translational part* of $\mathfrak{G}$, can be chosen so that

$$\mathfrak{G} = \mathfrak{a} + \mathfrak{v} \tag{21}$$

(direct sum of vector spaces). Any element of $\mathfrak{G}$ can then be resolved into two components, one in $\mathfrak{a}$ and one in $\mathfrak{v}$. For example, for the Maurer–Cartan form we write $\theta = a\theta + v\theta$. The vertical vectors are those that satisfy $(a\theta)\, X = 0$. Those that satisfy $(v\theta)\, X = 0$ will be called *θ-horizontal.* Any vector can be expressed as a sum of a θ-horizontal component and a vertical component, $X = aX + vX$. There is a θ-horizontal space at each point of G. The system of θ-horizontal spaces is invariant under left translations. The choice of a particular translational part $\mathfrak{a}$ of $\mathfrak{G}$ corresponds to choosing the θ-horizontal space at any one point of G.

Let $\{R_a\}$ be a basis for the algebra $\mathfrak{v}$ of fundamental fields. Then

$$[R_a, R_b] = c_{ab}{}^c R_c \tag{22}$$

where the $c_{ab}{}^c$ are the structure constants of H. A basis for the θ-horizontal space at any one point determines a set $\{R_\alpha\}$ of left-invariant θ-horizontal fields, providing a basis for $\mathfrak{a}$. The R_a and R_α together constitute a right vielbein R_A, where

$$A = (\alpha, a) \tag{23}$$

Referred to this basis of $\mathfrak{G}$, the two parts of θ are explicitly

$$a\theta = R^\alpha G_\alpha, \qquad v\theta = R^a G_a \tag{24}$$

and the θ-horizontal and vertical components of a vector X are explicitly

$$aX = X^\alpha R_\alpha, \qquad vX = X^a R_a \tag{25}$$

Since N preserves the fiber H, the vectors L_a given by (2) are vertical on H. Vertical vectors are those satisfying $R^\beta X = 0$. Therefore, on H, $R^\beta L_a = 0$. The matrices of the adjoint representation of G therefore satisfy

$$D_a{}^\beta(h) = 0 \tag{26}$$

for every $h \in H$. The submatrices $D_a{}^b(h)$ are those of the adjoint representation of H, and the matrices $D_\alpha{}^\beta(h)$ provide another representation of H that will be important in what follows. An alternative proof of (26) comes from noting that vertical spaces are mapped to vertical spaces by the right action of H (i.e., the right action of H preserves the fibers). Therefore $dR_h R_a$ is a linear combination of the R_b, for every $h \in H$. Comparison with the transformation law (15) then gives us (26) and also the behavior of the basis of $\mathfrak{v}$ under the right action of H

$$dR_h R_a = D_a{}^b(h)\, R_b \tag{27}$$

If the system of θ-horizontal spaces is invariant under the right action of H, the space G/H is called *reductive*. In this case, $dR_h R_\alpha = D_\alpha{}^\beta(h)\, R_\beta$ and $D_\alpha{}^b(h) = 0$. (Observe that the infinitesimal form of this condition, $[R_a, R_\beta] = c_{a\beta}{}^\gamma R_\gamma$, $c_{a\beta}{}^c = 0$, or $[\mathfrak{v}, \mathfrak{a}] \subseteq \mathfrak{a}$ is somewhat less restrictive if H is not a connected group; it implies only that the system of θ-horizontal spaces is invariant under the right action of H_0.) From (14) we see that $D_\alpha{}^b(h) = 0$ is equivalent to the condition that $D_\alpha{}^\beta(h)\, G_\beta = S^{-1}(h)\, G_\alpha S(h)$ for any representation S of G and for all $h \in H$. That is, $h^{-1}\mathfrak{a}h \subseteq \mathfrak{a}$ for all $h \in H$. The transformation law (10) then implies

$$R^*_{h^{-1}}(v\theta) = h(v\theta)\, h^{-1} \tag{28}$$

if and only if G/H is reductive. Therefore, the $\mathfrak{v}$-valued 1-form $v\theta$ is a connection form (in the usual sense) on the principal fiber bundle $G(G/H, H)$ if and only if G/H is reductive. Incidentally, its curvature form vanishes if and only if $\mathfrak{a}$ is a subalgebra of $\mathfrak{G}$. In what follows, we do *not* impose the restriction that G/H be reductive.

Let $\bar{S}$ be a matrix representation of H and let ρ be the vector space on which the matrices act. A mapping $\Psi: G \to \rho$ (i.e., a ρ-valued scalar field) is called *equivariant* if

$$R^*_{h^{-1}} \Psi = \bar{S}(h)\, \Psi \tag{29}$$

for every $h \in H$. This concept can be extended to ρ-valued vector fields; a ρ-valued vector field X is equivariant if, for every scalar Ψ invariant under the right action of H, $X(\Psi)$ is equivariant. It follows that X satisfies a condition of the form

$$dR_h X = \bar{S}(h)\, X \tag{30}$$

for every $h \in H$. A ρ-valued 1-form field ω is equivariant if $\omega(Y)$ is equivariant, for every right-invariant vector field Y. It then follows that $R^*_{h^{-1}}\omega = \bar{S}(h)\, \omega$. The concept of equivariance is readily extended to tensor fields. Conditions such as (29) or (30), satisfied by equivariant fields, are called *fiber conditions*. An important property of an equivariant field is that *it is determined on the whole of a fiber by its value at any one point of the fiber*. For example, (29) is equivalent to

$$\Psi(zh) = \bar{S}(h^{-1})\, \Psi(z) \tag{31}$$

The group G of left translations maps fibers to fibers and so induces, in an obvious way, a group of diffeomorphisms on G/H. We interpret G/H as space-time and we interpret pullbacks (from a section) of equivariant fields as fields on space-time. The subgroup of the group G of *left translations* that does not affect G/H corresponds to an internal symmetry group I. The subgroup I is the maximal subgroup of H that is an *invariant* subgroup of G. The group G/I is a space-time symmetry (Poincaré, affine or conformal group, with G/H Minkowski space, or de Sitter group with G/H de Sitter space). The whole group G is a semidirect product $(G/I) \times I$.

We now generalize the group G of left translations to a gauge group, and introduce the appropriate connection for this gauge group.

5. GAUGE TRANSFORMATIONS

We define a *gauge transformation* to be a diffeomorphism f on G that commutes with the right action of H

$$R_h f = f R_h \tag{32}$$

for every $h \in H$. Equivalently

$$f(zh) = f(z)\, h \tag{33}$$

for every $h \in H$. Observe that *the action of a gauge transformation on every point of a fiber is determined if its action on any one point of the fiber is given.* Define $g(z) = f(z)\, z^{-1}$. Then $g(zh) = g(z)$. Thus, associated with any gauge transformation f is a mapping $g: G/H \to G$ such that $f(z) = g(x)\, z$ $(x = \pi z)$. A gauge transformation is therefore a "gauged left translation" in the Yang–Mills sense that a group is generalized to a gauge group by allowing the group elements to be space-time-dependent. (Note, however, that not every G-valued function on G/H specifies a gauge transformation; in general, $f(z) = g(x)\, z$ is not even one-to-one.) Gauge transformations on a principal fiber bundle $P(M, H)$ were defined to be diffeomorphisms satisfying (33) by Atiyah, Hitchin, and Singer [9], but these authors explicitly restricted them to gauge transformations associated with an internal symmetry group, by imposing the additional condition $\pi f(z) = \pi z$ that no action be induced on M.

Let Y be a fundamental field and let R_h be an element of the one-parameter group of right translations generated by it. Then (21) implies, for any gauge transformation f and any vector field X, that $dR_h\, df X = df\, dR_h X$. For an infinitesimal H, this is $[Y, df X] = df[Y, X] = [df Y, df X]$. Since this is valid for any X, we have $Y = df Y$. Therefore, *fundamental fields are gauge-invariant.* If H is connected, the converse holds; any diffeomorphism that leaves every fundamental field invariant is a gauge transformation. If H is *not* connected we find that diffeomorphisms that leave every fundamental field invariant are somewhat more general than gauge transformations in that they need satisfy (32) or (33) only for $h \in H_0$, the connected subgroup of H. These more general diffeomorphisms have the form $f(z) = g(x)\, z$ where $g(x)$ is multiple-valued, depending on the piece of the fiber $\pi^{-1}x$ in which z is located as well as on x. These diffeomorphisms leaving fundamental fields invariant are in fact the gauge transformations of $G(G/H_0, H_0)$, rather than of $G(G/H, H)$.

Let Λ be a generator of a one-parameter group of gauge transformations. Then, for an infinitesimal gauge transformation generated by Λ, (32) implies $[\Lambda, dR_h Y] = dR_h[\Lambda, Y] = [dR_h \Lambda, dR_h Y]$ for *any* vector field Y. Therefore, $\Lambda = dR_h \Lambda$ for any $h \in H$. *The generators of gauge transformations are the vector fields that are invariant under the right action of H.*

Applying a gauge transformation f to the fiber condition (29), we find that $f^* \Psi$ satisfies the same fiber condition as Ψ (and similarly for eq. 30). That is, *equivariance is a gauge-invariant property*.

The gauge transformations that do not induce diffeomorphisms on G/H (more precisely, the gauge transformations that induce the trivial diffeomorphism on G/H) are of particular interest. They form a larger group than the group of "gauged internal symmetries." These "*pure*" *gauge transformations* satisfy the extra constraint

$$\pi f(z) = \pi z \tag{34}$$

Since z and $f(z)$ are in this case on the same fiber, $\eta(z) = z^{-1} f(z) \in H$ for every $z \in G$. The H-valued mapping $\eta : G \to H$ is equivariant, with fiber condition

$$\eta(zh) = h^{-1} \eta(z)\, h \tag{35}$$

The correspondence between pure gauge transformations and these equivariant H-valued mappings is a group isomorphism $[z^{-1} f_1 \circ f_2(z) = z^{-1} f_1(z) \cdot z^{-1} f_2(z)$ where $\cdot$ is the group multiplication in H.] Under a pure gauge transformation f

$$f(z) = z\eta(z) \tag{36}$$

the transformation law of the Maurer–Cartan form (9) is easily seen to be

$$f^{*-1}\theta = \eta\theta\eta^{-1} - d\eta \cdot \eta^{-1} \tag{37}$$

6. GENERALIZED CONNECTIONS

We define a *generalized connection* on $G(G/H, H)$ to be a $\mathfrak{G}$-valued 1-form ω satisfying

$$\begin{aligned} &1. \quad R^*_{h^{-1}}\omega = h\omega h^{-1} \\ &2. \quad \omega(X)=0 \text{ implies } X=0 \\ &3. \quad \text{For any } \textit{vertical} \text{ vector } X,\ \omega(X)=\theta(X) \end{aligned} \tag{38}$$

That is, ω is equivariant, satisfying the same fiber condition as the Maurer–Cartan form; ω is nonsingular and maps any vertical vector to the fundamental field to which it belongs. In particular, the Maurer–Cartan form is a generalized connection, called here the *trivial* connection.

Let ω be a connection (we now drop the qualifying adjective generalized) and write $\omega = E^A G_A$. Splitting ω into a part belonging to $\mathfrak{a}$ and a part belonging to $\mathfrak{v}$

$$\begin{aligned} &\omega = a\omega + v\omega \\ &a\omega = E^\alpha G_\alpha, \qquad v\omega = E^a G_a \end{aligned} \tag{39}$$

Condition (3) implies, for the associated vielbein $\{E_A\}$

$$E_a = R_a \tag{40}$$

and condition (1) implies the fiber condition

$$dR_h E_A = D_A{}^B(h)\, E_B \tag{41}$$

The vectors satisfying $(a\omega)\,X=0$ are the vertical vectors. Vectors satisfying $(v\omega)\,X=0$ are called *ω-horizontal*, or simply *horizontal* vectors. Any vector can be expressed as a sum of a horizontal and vertical component

$$\begin{aligned} &X = aX + vX \\ &aX = X^\alpha E_\alpha, \qquad vX = X^a R_a \end{aligned} \tag{42}$$

(these projection operators a and v are not to be confused with the previous ones (25), which relate to the particular case $\omega=\theta$).

The connection ω specifies a system of horizontal spaces; the transformation law (41) shows that this system is invariant under the right action of H if and only if $D_\alpha{}^b(h)=0$, that is, if and only if G/H is reductive. In that case, $v\omega$ is a "connection in the usual sense" on $G(G/H, H)$, with transformation law $R_{h^{-1}}(v\omega) = h(v\omega)\,h^{-1}$.

Considering an infinitesimal right translation, one can deduce from (41) that $[R_a, E_B] = c_{aB}{}^C E_C$. Therefore, the scalar fields $\Omega_{AB}{}^C$ defined by

$$[E_A, E_B] = \Omega_{AB}{}^C E_C \tag{43}$$

satisfy $\Omega_{aB}{}^C = c_{aB}{}^C$. The *curvature* of a connection ω is the $\mathfrak{G}$-valued two-form

$$\mathcal{G} = -2(d\omega + \omega \wedge \omega) \tag{44}$$

Now, note that $2d\omega(E_A, E_B) = E_A(\omega E_B) - E_B(\omega E_A) - \omega[E_A, E_B] = -\Omega_{AB}{}^C G_C$ (we have made use of $\omega E_B = G_B$, and its consequence $E_A(\omega E_B) = 0$). Also, $2\omega \wedge \omega(E_A, E_B) = [\omega E_A, \omega E_B] = c_{AB}{}^C G_C$. Therefore

$$\mathcal{G}(E_A, E_B) = (\Omega_{AB}{}^C - c_{AB}{}^C)\, G_C \tag{45}$$

It follows that $\mathcal{G}(E_a, E_B) = 0$ and hence that $\mathcal{G}(X, Y) = 0$ if either X or Y or both is *vertical*. In other words

$$\mathcal{G}(X, Y) = \mathcal{G}(aX, aY) \tag{46}$$

We define a *fiber neighborhood* of $G(G/H, H)$ to be an open set $U = \pi^{-1} V$ where V is a neighborhood of G/H. Then, if U is a fiber neighborhood, we define a *local gauge transformation* with support U to be a diffeomorphic mapping $f\colon U \to fU$ satisfying $R_h f = f R_h$ (for every $h \in H$). From the theorems established in Section 3, we know that local diffeomorphisms f exist, supported on a neighborhood of any point, such that $\omega = f^*\theta$, if and only if $\mathcal{G} = 0$. Let f_0 be such a local diffeomorphism, with support U_0 (a neighborhood of some point in G). Then, from (40), we see that f_0 maps every fundamental field on U_0 to a fundamental field on $f_0 U_0$. The diffeomorphism f_0 satisfies $f_0(zh) = f_0(z)\, h$ for every z and zh in U_0. We can extend the support of f_0 to a fiber neighborhood $U = \pi^{-1}\pi U_0$. That is, there is a unique local diffeomorphism f with support U defined by $f(zh) = f(z)\, h$ (for every $z \in U$ and every $h \in H$) and the requirement that f coincides with f_0 on U_0. This establishes that, if ω is a connection, *there is a fiber neighborhood of every fiber and a local gauge transformation supported on it satisfying* $\omega = f^*\theta$, *if and only if* $\mathcal{G} = 0$.

The conventional fiber bundle geometry suggests an alternative way of defining a curvature. We define the pseudocurvature $\mathcal{F}$ of a connection ω to be the 2-form such that, for any two vectors X and Y

$$\mathcal{F}(X, Y) = -2d\omega(aX, aY) \tag{47}$$

From (44) and (46) it follows that

$$\mathcal{F} = \mathcal{G} + 2a\omega \wedge a\omega \tag{48}$$

Since $2(a\omega \wedge a\omega)(E_A, E_B) = \delta_A^\alpha \delta_B^\beta c_{\alpha\beta}{}^C G_C$, we have $\mathscr{F} = \mathscr{G}$ if and only if $\mathfrak{a}$ is an *abelian subalgebra* of $\mathfrak{G}$.

The transformation law of a connection ω under a *pure* gauge transformation f, $f(z) = z\eta(z)$, is

$$f^{*-1}\omega = \eta\omega\eta^{-1} - d\eta \cdot \eta^{-1} \tag{49}$$

We prove this by considering the case of an infinitesimal pure gauge transformation. A vector field Λ that generates gauge transformations is invariant under the right action of H, i.e., $[R_a, \Lambda] = 0$. It generates pure gauge transformations if it is *vertical*, $\Lambda = -\eta^a R_a$. This is just the infinitesimal expression of (36). The condition $[R_a, \Lambda] = 0$ implies $R_a \eta^b + \eta^c C_{ac}{}^b = 0$, which is the infinitesimal version of the fiber condition (35). We can write $\Lambda = -\eta^A E_A$, $\eta^\alpha = 0$ and consider the infinitesimal transformation law $\delta E_A = [\Lambda, E_A]$ of the vielbein. It is not difficult to deduce from this that $\delta E^A = -\delta\eta^A + \eta^B E^C \Omega_{BC}{}^A$. Since $\eta^\beta = 0$ and $\Omega_{bC}{}^A = c_{bC}{}^A$, we have $\delta E^A = -d\eta^A + \eta^B E^C c_{BC}{}^A$. This transformation law for the components of ω under an infinitesimal pure gauge transformation is just the infinitesimal form of (49).

7. FIELDS ON SPACE-TIME

Let $\sigma: G/H \to G$ be a section on the fiber bundle $G(G/H, H)$. If no global section exists, it is possible to work with a collection of local sections. (Details are not gone into here because the notation becomes cumbersome.) With the aid of σ, it is possible to define fields on space-time G/H as pullbacks of equivariant fields on G. If ω is a connection, we define the *gauge potential* to be the $\mathfrak{G}$-valued 1-form

$$\Gamma = \sigma^* \omega \tag{50}$$

on G/H. The $\mathfrak{a}$-valued 1-form $a\Gamma$ on G/H is *nonsingular*. To see this, suppose X is a vector on G/H satisfying $a\Gamma(X) = 0$. That is, $0 = (\sigma^* a\omega)X = (a\omega)(d\sigma X)$. Therefore, $d\sigma X$ is vertical. Now $d\pi$ maps vertical vectors to zero, so $d\pi\, d\sigma X = 0$, and since $\pi\sigma = 1$, $d\pi\, d\sigma$ is the unit mapping on the tangent spaces to G/H. Therefore, $X = 0$. Since the $\mathfrak{a}$-valued 1-form

$$e = a\Gamma = e^\alpha G_\alpha \tag{51}$$

is nonsingular, the 1-forms e^α on G/H are linearly independent. They provide a basis $\{e^\alpha\}$ for the cotangent spaces to G/H, and there is, dually, a vielbein $\{e_\alpha\}$ satisfying $e^\alpha(e_\beta) = \delta^\alpha_\beta$. Since G/H is interpreted here as space-time, e_α is a *tetrad*.

A connection ω is completely determined by its gauge potential Γ: Suppose ω and ω' are two connections with the same Γ. Then $\sigma^*\phi = 0$ where $\phi = \omega - \omega'$. Therefore $\phi(d\sigma X) = 0$ for any vector X of G/H. Therefore ϕ annihilates any vertical vector, because both ω and ω' satisfy (38, 40). Since any vector on G at a point on $\sigma(G/H)$ can be resolved into a vertical component and a component tangential to $\sigma(G/H)$, it follows that $\phi = 0$ on $\sigma(G/H)$. But ϕ is equivariant, and therefore $\phi = 0$ everywhere.

The diffeomorphism φ induced on G/H by a gauge transformation f is $\varphi = \pi f \pi^{-1}$, or alternatively

$$\varphi = \pi f \sigma \tag{52}$$

(although it is, of course, independent of the choice of σ). We denote the points of G/H by x and write $x' = \varphi(x)$. With reference to a given section σ, we can associate, with any gauge transformation f, a unique H-valued field h on G/H, i.e., a mapping $h: G/H \to H$, defined by

$$h(x) = [\sigma(x')]^{-1} f[\sigma(x)] \tag{53}$$

Every gauge transformation specifies in this way a unique pair of mappings $\varphi: G/H \to G/H$ and $h: G/H \to H$. Conversely, the pair of mappings φ, h uniquely specifies the gauge transformation f (this is fairly obvious from the geometrical situation). We accordingly introduce the notation $f = (\varphi, h)$. We have already defined a *pure* gauge transformation to be one for which φ is trivial ($\varphi = 1$). We define a *special* gauge transformation to be one for which h is trivial [$h(x) = e$ for every $x \in G/H$, which we denote by $h = 1$]. The product rule for gauge transformations, in terms of this decomposition, is $(\varphi_1, h_1) \circ (\varphi_2, h_2) = (\varphi, h)$ where $\varphi = \varphi_1 \circ \varphi_2$, and $h(x) = h_1(\varphi_2 x) \cdot h_2(x)$. Consequently, any gauge transformation can be regarded as a product of a *pure* gauge transformation followed by a *special* gauge transformation: $(\varphi, h) = (\varphi, 1) \circ (1, h)$. Observe that, for a pure gauge transformation $f(z) = z\eta(z)$

$$h(x) = \eta[\sigma(x)] \tag{54}$$

(i.e., $h = \sigma^*\eta$ or $h = \eta \circ \sigma$).

Under a special gauge transformation, Γ transforms simply as a 1-form under the diffeomorphism φ. Under a pure gauge transformation, the transformation law of Γ follows from the transformation law (49) of ω in conjunction with (54). The transformation law of Γ under a gauge transformation $f = (\varphi, h)$ then turns out to be

$$\Gamma' = \sigma^*(f^{-1}\omega) = \varphi^{*-1}(h\Gamma h^{-1} - dh \cdot h^{-1}) \tag{55}$$

In particular, the *tetrad* $\{e_\alpha\}$ has the homogeneous transformation law

$$e'_\alpha = d\varphi[D_\alpha{}^\beta(h)\, e_\beta] \tag{56}$$

The *gauge field* associated with the gauge potential Γ is the pullback of the curvature, $G = \sigma^* \mathscr{G}$. That is

$$G = -2(d\Gamma + \Gamma \wedge \Gamma) \tag{57}$$

Its transformation law under a gauge transformation is homogeneous

$$G' = \varphi^{*-1}(hGh^{-1}) \tag{58}$$

Its components define a curvature G^a and a torsion G^α.

The "H-curvature" and "H-torsion" are similarly defined from the pullback $F = \sigma^* \mathscr{F}$. The anholonomic components of this $\mathfrak{G}$-valued 2-form on G/H are $F_{\alpha\beta}{}^C = F^C(e_\alpha, e_\beta) = (\sigma^* \mathscr{F}^C)(d\sigma E_\alpha, d\sigma E_\beta) = \sigma^*[(\mathscr{F}^C(E_\alpha, E_\beta)] = \sigma^* \Omega_{\alpha\beta}{}^C = \Omega_{\alpha\beta}{}^C(\sigma)$.

$$F_{\alpha\beta}{}^C = \Omega_{\alpha\beta}{}^C(\sigma) \tag{59}$$

Equation (48) tells us that

$$F = G + 2e \wedge e \tag{60}$$

from which it follows that $F = G$ if and only if $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{G}$ (which is of course the case for the Poincaré, affine, and conformal groups, but not for the de Sitter group. Incidentally, note that the de Sitter gauge theories will also have additional complications in our scheme because a global section σ does not exist in this case and we are forced to interpret σ as a collection of local sections).

A "matter field" on the space-time G/H can be interpreted as the pullback of an equivariant scalar field (generalization to equivariant p forms is straightforward)

$$\psi = \sigma^* \Psi = \Psi \circ \sigma \tag{61}$$

Its transformation law under a gauge transformation turns out to be

$$\psi' = \varphi^{*-1} \bar{S}(h)\, \psi \tag{62}$$

The infinitesimal form of this transformation law (62) is of interest. It enables us to show clearly the relationship between the formalism of this work and that of the standard approach to Poincaré gauge theories [10, 11]. Let Λ be an infinitesimal vector field, invariant under the right action of H and therefore generating an infinitesimal gauge transformation. The infinitesimal transformation law of a scalar field Ψ on G is simply

$$\delta\Psi = \Lambda(\Psi) \tag{63}$$

The anholonomic components $\Lambda^A = E^A(\Lambda)$, pulled back, give a set of scalars $\lambda^A = \sigma^* \Lambda^A = \Lambda^A \circ \sigma$ on G/H which (because of the equivariance of Λ), completely determine Λ. They are to be regarded as the (space-time-dependent) parameters of the infinitesimal gauge transformation. The action of the infinitesimal gauge transformation on the field ψ is

$$\delta\psi = \lambda^A Q_A \psi, \qquad Q_A \psi = \sigma^* E_A \psi \tag{64}$$

The operators Q_A are the generators of the action of gauge transformations on ψ and are well-defined operators when acting on the pullback of any equivariant field. Their commutation relations are easily obtained from (43) and (64). We find $[Q_A, Q_B] = \Omega_{AB}{}^C(\sigma)\, Q_C$ or, in terms of the H curvature and H torsion

$$\begin{aligned} [Q_\alpha, Q_\beta] &= F_{\alpha\beta}{}^C Q_C \\ [Q_a Q_B] &= c_{aB}{}^C Q_C \end{aligned} \tag{65}$$

The quantity $Q_A \psi$ transforms homogeneously under a gauge transformation $f = (\varphi, h)$

$$(Q_A \psi)' = \varphi^{*-1} D_A{}^B(h)\, \bar{S}(h)\, Q_B \psi \tag{66}$$

and so can be regarded as a *covariant derivative* of ψ. Explicit expressions for $Q_A \psi$ are

$$\begin{aligned} &Q_\alpha \psi = D_\alpha \psi = (D\psi)\, e_\alpha, \qquad D\psi = d\psi + \Gamma^a \bar{G}_a \psi \\ &Q_a \psi = -\bar{G}_a \psi \end{aligned} \tag{67}$$

where the $\bar{G}_a$ are the generators of the matrix representation $\bar{S}$ of H. The reader is referred to Ref. 8 for further details (but beware of the sign conventions, which are different from those employed here).

8. CONCLUDING REMARKS

Existing attempts at constructing gauge theories of space-time symmetries are many and varied; see, for example, the review article of Ivanenko and Sardanashvily [3] and the works cited therein. Our aim has been to construct a clear and consistent geometrical background that avoids the pitfalls and complications of many earlier attempts. Questions of Lagrangian structure have not been touched upon; a Lagrangian would be a gauge-invariant constructed from ω and Ψ, the independent dynamical fields being the Γ and ψ. Ne'eman and Regge [7] and Pérez–Rendon and Ruiperez [12] have investigated the problem of Lagrangian structure in a scheme that is similar to ours.

Many existing fiber-bundle descriptions of gauged space-time symmetries run into difficulties and complications by attempting a too close analogy with the standard fiber-bundle descriptions of *internal* symmetries; attempts are made to have the *whole* of the group G acting on the fibers. Various *ad hoc* concepts then have to be introduced (affine frames, second-order frames, soldering forms, etc.) and the identification between a translational gauge potential and a tetrad does not work out [3, 4]. In our view, in a fruitful approach to a fiber-bundle description of a gauged space-time group G, *only the subgroup H* should act on the fiber. "Internal translations" are not needed. The above-mentioned difficulties and complications are then not encountered. In the scheme we have presented, the translational gauge potential is essentially a tetrad and the linear independence of the tetrad vectors is ensured.

In their action on space-time (base-space G/H) and on fields over space-time, the gauge transformations consist of a local action of H together with general diffeomorphisms of space-time. By a formal reinterpretation, "general diffeomorphisms" could be replaced by "general coordinate transformations." A careful analysis (carried out by von der Heyde for the Poincaré case [10]) reveals that it is more proper to regard the action of a "gauged translation" as a parallel transport operation rather than a general coordinate transformation. This interpretation is implicit in our equation (67) for Q_α. Note that $Q_\alpha \psi = D_\alpha \psi$ does not transforms homogeneously and independently of $Q_a \psi$, if G/H is not reductive (e.g., conformal gauge theory [13]).

The interpretation of a gauged generalization of H, together with space-time diffeomorphisms, as a gauged generalization of a space-time group G, is *fully justified* by the fact that, in the "ungauged limit" (group G of left translations), we recover a realization of the group G as a group of actions on the points x of space-time and on space-time fields ψ. Indeed, we get the "correct" action (the usual action of the Poincaré group as the group of isometries of Minkowski space, action of the conformal group on Minkowski space, action of the de Sitter group as the group of isometries of de Sitter space). The realization of the group G as a group of transformations on points x of space-time and fields ψ over space-time is in fact an example of a nonlinear realization in the sense of Coleman, Wess, and Zumino [14] or Salam and Strathdee [15]. Space-time itself takes the place of Goldstone fields. The "trivial" potential $\mathring{\Gamma} = \sigma^*\theta$ appropriate to the ungauged (global) action of G is $\sigma^{-1}\,d\sigma$, which is analogous to the connection of Callan *et al.* [16]; its two pieces $a\mathring{\Gamma}$ and $v\mathring{\Gamma}$ are analogous to their p and v.

The Poincaré version of the relations (65), with curvature and torsion taking the place of structure constants [11] have occasionally been criticized [3, 17]. The view taken seems to be that, since the generators Q_A do not satisfy the commutation relations of $\mathfrak{G}$, there must be something wrong in interpreting the transformations generated by them as a gauged generalization of the action of G. The situation can be clarified by considering the freedom associated with the *choice of parametrization* of the infinitesimal gauge group. The generators Q_A arose from choosing as parameters the pullbacks of the anholonomic components of Λ with respect to the vielbein E_A. If, instead, we employ the left vielbein $\{L_A\}$ for this purpose, the parameters are $a^A = \sigma^*(L^A\Lambda)$, (53) becomes $\delta\psi = a^A M_A\psi$, $M_A\psi = \sigma^* L_A\psi$, and the generators $-M_A$ *satisfy the usual commutation relations*, $[M_A, M_B] = -c_{AB}{}^C M_C$ (compare eq. (2.11) of Harnad and Pettitt [18]). A further pleasant feature of this parametrization is that the parameters for the infinitesimal global transformations (left translations) are *constants*. However, the generators Q_A are more appropriate than the M_A in the context of the gauged generalization, because of the simple transformation law (66) of the $Q_A\psi$, and because of the way that the interpretation of gauged translations as parallel transport is manifested in (67).

The constraints (38) distinguish our generalized connection ω from the more general ρ of Ne'eman and Regge. Our aim has been to provide a consistent fiber-bundle description for a certain class of *classical* physical theories (Hehl's Poincaré gauge theory and its generalizations, with or without internal symmetries). In this context, the constraints (38) are just what are needed to produce the gauge potentials in the base space (as pullbacks). The relaxation of the third constraint becomes desirable in a quantized theory, since the degrees of freedom that are suppressed by this constraint are the ones that correspond to the ghosts [19, 20].

The generalization of our scheme to incorporate supersymmetric theories appears to be straightforward. The algebra $\mathfrak{G}$ is in this case a graded Lie algebra. The fermionic generators of supersymmetry are included in the translational part $\mathfrak{a}$. The base space G/H is replaced by a superspace. Equation (67) incorporates the action of gauged supersymmetry as well as gauged ordinary translations. Ne'eman and Regge identified gauged supersymmetries as anholonomized general coordinate transformations. Our gauge transformations are diffeomorphisms on the bundle G; interpreted passively, they are coordinate transformations on G, and our interpretation of supersymmetry is then identical to theirs.

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