Gauge theory of a group of diffeomorphisms. III. The fiber bundle description

Eric A. Lord
Department of Applied Mathematics, Indian Institute of Science, Bangalore 560 012, India
P. Goswami
Department of Physics, Indian Institute of Science, Bangalore 560 012, India

(Received 17 June 1986; accepted for publication 5 August 1987)

A new fiber bundle approach to the gauge theory of a group G that involves space-time symmetries as well as internal symmetries is presented. The ungauged group G is regarded as the group of left translations on a fiber bundle $G(G/H,H)$, where H is a closed subgroup and G/H is space-time. The Yang–Mills potential is the pullback of the Maurer–Cartan form and the Yang–Mills fields are zero. More general diffeomorphisms on the bundle space are then identified as the appropriate gauged generalizations of the left translations, and the Yang–Mills potential is identified as the pullback of the dual of a certain kind of vielbein on the group manifold. The Yang–Mills fields include a torsion on space-time.

I. INTRODUCTION

The exploitation of the structures known as fiber bundles,[1,2] for the formulation of Yang–Mills theories (gauge theories), has received a great deal of attention in recent years. In the conventional approach to the gauging of a symmetry group G, one puts a connection (Lie algebra valued one-form) on a principal fiber bundle $P(M,G)$ (M being space-time) and interprets a "gauge transformation" as a change of section in this bundle.[3–6] There are indications that this approach is not an appropriate one if G involves space-time symmetries. Consider, for example, Poincaré gauge theories,[7–10] affine gauge theories,[11–14] and conformal gauge theories.[15,16] The conventional fiber bundle descriptions of these theories employ *ad hoc* structures: second-order frames[16] (conformal) and affine frames[1–17] (Poincaré and affine).The translational gauge potentials on M and the tetrad on M turn out to be conceptually distinct entities, as has been pointed out by several authors[6,17]; this is a clear indication of the inappropriateness of the conventional fiber bundle description, in the case of space-time symmetries. In particular, the conventional fiber bundle description is not appropriate for a Poincaré gauge theory.

We present an alternative fiber bundle description of gauge theories that does not encounter the difficulties mentioned above, and that provides a unified scheme for describing space-time and internal symmetries, and their "gauged" generalizations.

Our approach is based on the properties of the fiber bundles $G(G/H,H)$, where G is a Lie group, H a Lie subgroup, and G/H is interpreted as space-time. The group to be gauged is the group G of left translations and a gauge transformation is a bundle automorphism. A connection will be defined essentially as a particular kind of vielbein on G.

The idea of formulating gauge theories on a principal fiber bundle $G(G/H,H)$, with G/H interpreted as space-time, is not new. It was proposed and investigated by Ne'eman and Regge,[18,19] who studied the problem of constructing Lagrangians on the bundle space and showed that the idea can be extended consistently to provide a framework for supergravity theories. Our scheme differs from theirs in several important respects; their concept of gauge transformation was different from ours and their connection vielbein was not specialized. Further investigations into the construction of Lagrangians in the scheme of Ne'eman and Regge have been made by Pérez-Rendon and Ruiperez.[20] In the present work we shall not consider the problem of constructing Lagrangian theories; our emphasis is on the geometrical structure only.

In recent work[21] we have shown how Poincaré gauge theory can be generalized to groups other than the Poincaré group (such as the affine, de Sitter, and conformal groups). The fiber bundle concept was not employed—the formalism dealt only with quantities defined as fields on space-time. The present work is a fiber bundle interpretation of these ideas.

II. THE PRINCIPAL FIBER BUNDLE $G(G/H,H)$

Let G be a Lie group and H a closed Lie subgroup. The orbits of the *right* action of H on G are the left cosets gH. They are the fibers of the principal fiber bundle $G(G/H,H)$ whose structural group is H (acting on the right). Denoting the general element of G by z, the *left translation* associated with an element of $g\in G$ is the diffeomorphism $z\to z'=gz$ on the group manifold. On account of the associative law, the left translations constitute a group of diffeomorphisms on G, isomorphic to G. Throughout most of this work, we impose none of the common restrictions on G (such as semisimplicity, compactness, and connectedness). We find that they are not necessary.

Let σ: $G/H\to G$ be a section on $G(G/H,H)$. (If no global section exists, σ can be a collection of local sections; the notation for dealing with this case becomes cumbersome but the principles we shall develop remain valid. For simplicity, we shall not enter into these details.) Then any element $z\in G$ can be uniquely expressed as a product

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$$z=\sigma(x)\chi\,, \tag{2.1}$$

with $x\in G/H$, $\chi\in H$ (the point $x\in G/H$ is just πz, where π is the canonical projection on the bundle).

Left translations map fibers to fibers and so induce, in an obvious way, diffeomorphisms on G/H. This enables us to associate, with any left translation $z'=gz$, an H-valued field $h(g,x)$ on G/H, defined by

$$g\sigma(x)=\sigma(x')h(g,x) \tag{2.2}$$

(where x' denotes the image of $x\in G/H$ under the diffeomorphism induced on G/H by $z\to z'=gz$). The geometrical interpretation of (2.2) is illustrated in Fig. 1, in which it is to be understood that g acts on the left and $h=h(g,x)$ on the right.

We shall use a prime to denote the transform of a tensor field under the action of a diffeomorphism. Thus the action of the diffeomorphism $z\to z'=f(z)$ on a p-form field $\phi=(f^{-1})^*\phi$ and the action on a vector field V will be written $V'=(df)V$. Let Ψ be a set of p-forms that transform linearly among themselves according to some matrix representation S of H, under the right action of H:

$$\Psi'=\overline{S}(h)\Psi \quad (z'=zh)\,. \tag{2.3}$$

In this definition Ψ is a pseudotensorial form of type $(\overline{S},\mathfrak{h})$, where $\mathfrak{h}$ is the Lie algebra of H. In the case of scalar fields, (zero-forms) this prescription is equivalent to the usual construction of an associated fiber bundle.[3] We shall refer to a condition of the form (2.3) as a *fiber condition*. A set of fields Ψ satisfying a fiber condition is determined on the whole of a fiber if its value at one point of the fiber is given. For a set Ψ of *scalar* fields, the fiber condition (2.3) can be written in the alternative form

$$\Psi(zh)=\overline{S}^{-1}(h)\Psi(z)\,. \tag{2.4}$$

In the case of a set Ψ of p-forms, a set ψ of p-forms on G/H can be defined as the pullback

$$\psi=\sigma^*\Psi\,. \tag{2.5}$$

For a set Ψ of *scalar* fields, this is simply

$$\psi(x)=\Psi(\sigma(x)) \tag{2.6}$$

and the fiber condition (2.3) then leads to the transformation law

$$\psi(x')=\overline{S}(h(g,x))\psi(x) \tag{2.7}$$

under the action of a *left translation* $z'=gz$.

Equations (2.2) and (2.7) are essentially the fundamental relations of the theory of nonlinear realizations.[22–24]

In the present interpretation, G/H is a space-time. Then (2.7) is the active transformation law of a set of physical fields under the action of both space-time symmetries and internal symmetries. The subgroup I whose left action does not affect the points of G/H is the maximal subgroup of H that is an invariant subgroup of G. The group of diffeomorphisms on G/H induced by the left action of G is isomorphic to G/I.[2] The group I can be interpreted as an internal symmetry group and G/I as a space-time symmetry group. The reasonable candidates for G/I are the Poincaré group (with G/H Minkowski space), the conformal group (with G/H Minkowski space, or, in the case of the "full" conformal group including inversions, Minkowski space augmented by a "light cone at infinity"[25]), the de Sitter group[26,27] (acting on de Sitter space G/H), and the affine group (with G/H a metricless four-space which can, however, in a dynamical theory, acquire a metric as a Goldstone field when the affine symmetry is broken[28,29]).

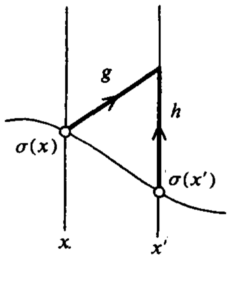

FIG. 1. The geometrical interpretation of (2.2).

III. DIFFERENTIAL GEOMETRY OF A GROUP MANIFOLD

In order to proceed further, we shall need to review briefly the elementary concepts from the differential geometry of a Lie group. The reader will almost certainly be familiar with these concepts, which are well known. However, our method of presentation is somewhat novel; it is aimed at establishing the results we shall need in the most rapid possible way.

Denote the general point of the manifold of a Lie group G by z. Denote an element of G regarded as an operator that acts on the group manifold (by left or right multiplication) by g. Of course, every element is simultaneously a point of the manifold and an operator on the manifold, but the conceptual distinction is, nevertheless, a useful one. Let e denote the unit element. Put a coordinate system on the group manifold and use the letters $M,N,\ldots$ for holonomic indices (so that the coordinates of z will be written z^M). Call this the "main" coordinate system. In addition, introduce an extra coordinate chart U, containing e, and use the letters $A,B,\ldots$ for coordinate labels in U.

In terms of coordinates, an infinitesimal left translation $z'=gz$ is given by $g^A=e^A-a^A$ (a^A infinitesimal), $z'^M=(gz)^M=z^M-a^AL_A{}^M$, where

$$L_A{}^M(z)=\left.\frac{\partial(gz)^M}{\partial g^A}\right|_{g=e}. \tag{3.1}$$

Write L for the matrix whose matrix elements are $L_A{}^M$ and write $L_M{}^A$ for the matrix elements of L^{-1}. The quantities $L_A{}^M$ are the components of a vielbein (linear frame) on the group manifold, which we shall call the *left vielbein*. The vector fields $L_A=L_A{}^M\partial_M$ ($\partial_M=\partial/\partial z^M$) provide a basis for the tangent space to G at each point, and the dual basis for the contangent space is provided by the one-forms $L^A=dz^M\,L_M{}^A$.

Under a change of the main coordinate system, $L'^M_A(z')=L_A{}^N(z)\partial z'^M/\partial z^N$. Under a change of the coordinatization on U, $L'_A{}^M=K_A{}^BL_B{}^M$, where K is a constant matrix ($K_A{}^B=\partial g^B/\partial g'^A|_{g=e}$). This corresponds to a change of basis of the Lie algebra (see below). The indices $A,B,\ldots$ have taken on the role of anholonomic indices.

In an exactly analagous manner, we can define

$$R_A{}^M(z) = \left.\frac{\partial(zg)^M}{\partial g^A}\right|_{g=e} . \tag{3.2}$$

Write R for the matrix whose matrix elements are $R_A{}^M$ and write $R_M{}^A$ for the matrix elements of R^{-1}. We have a *right vielbein* consisting of the vector fields $R_A = R_A{}^M \partial_M$ and the dually related one-forms $R^A = dz^M R_M{}^A$.

A left-invariant vector field X is one for which $X' = X$ under any left translation. The left-invariant vector fields are the linear combinations of the vector fields R_A, with constant coefficients. In particular, the vector fields R_A are left invariant, so $R_A{}^M(z') = R_A{}^N(z)\partial z'^M/\partial z^N$ $(z' = gz)$. Therefore, for a left translation,

$$\frac{\partial z'^M}{\partial z^N} = [R^{-1}(z)R(z')]_N{}^M \quad (z' = gz) . \tag{3.3}$$

Similarly, right-invariant vector fields are linear combinations of L_A, and, for a right translation,

$$\frac{\partial z'^M}{\partial z^N} = [L^{-1}(z)L(z')]_N{}^M \quad (z' = zg) . \tag{3.4}$$

Now let S be a matrix field on G, providing a matrix representation of G. The *generators* of the representation are the matrices

$$G_A = \left.\frac{\partial S(g)}{\partial g^A}\right|_{g=e} . \tag{3.5}$$

Differentiating $S(gz) = S(g)S(z)$ with respect to g^A and setting $g = e$, we get

$$L_A S = G_A S . \tag{3.6}$$

Similarly,

$$R_A S = S G_A . \tag{3.7}$$

Hence

$$S^{-1} G_A S = D_A{}^B G_B , \tag{3.8}$$

where D is the matrix field

$$D = LR^{-1} . \tag{3.9}$$

It is obvious from (3.8) that the matrices D provide a representation of G. It is of course the *adjoint representation*. Write c_A for its generators and define the structure constants of G in terms of the matrix elements of the c_A:

$$c_{AB}{}^C = (c_B)_A{}^C . \tag{3.10}$$

Now write (3.8) in the form $G_A S(g) = D_A{}^C(g) S(g) G_C$, differentiate with respect to g^B and set $g = e$, and we obtain the familiar commutation relations

$$[G_A, G_B] = c_{AB}{}^C G_C . \tag{3.11}$$

From (3.6) and (3.7) we now readily derive the commutation relations satisfied by the left and right vielbeins:

$$[R_A, R_B] = c_{AB}{}^C R_C , \tag{3.12}$$

$$[L_A, L_B] = -c_{AB}{}^C L_C , \tag{3.13}$$

$$[R_A, L_B] = 0 . \tag{3.14}$$

These equations are the infinitesimal forms of the transformation laws of the left and right vielbeins under left and right translations. The finite forms are

$$L'_A = D_A{}^B(g^{-1}) L_B, \quad R'_A = R_A \quad (z' = gz) ; \tag{3.15}$$

$$L'_A = L_A, \quad R'_A = D_A{}^B(g) R_B \quad (z' = zg) . \tag{3.16}$$

(For example, under the left translation $z' = gz$, $L'_A{}^M(z') = L_A{}^N(z) \quad [R^{-1}(z)R(z')]_N{}^M$, and $L(z)R^{-1}(z)R(z') = D(z)D^{-1}(z')L(z') = D^{-1}(g)L(z')$. This establishes the first of the above transformation laws. The others are proved similarly.)

The algebra of left-invariant vector fields, defined through the commutator product rule (3.12), is the *Lie algebra* of G. Because of (3.11), the generators of any matrix representation of G provide a matrix representation of the Lie algebra.

The $R_M{}^A$ are the components of the *Maurer–Cartan form*. This is the Lie-algebra valued one-form θ which, in any matrix representation of the Lie algebra with generators G_A, is represented by $R^A G_A$. From (3.7) we have $\partial_M S = R_M{}^A S G_A$ and therefore $R_M{}^A G_A = S^{-1}\partial_M S$. This can be written in a more abstract form, without reference to any particular representation, simply as

$$\theta = z^{-1} dz . \tag{3.17}$$

From (3.15) and (3.16) it follows that

$$\begin{aligned} R'^A &= R^A \quad (z' = gz) , \\ R'^A &= R^B D_B{}^A(g^{-1}) \quad (z' = zg) , \end{aligned} \tag{3.18}$$

which mean that the Maurer–Cartan form is left invariant and transforms under right translations according to the coadjoint representation $(D^T)^{-1}$. With the aid of (3.8) this behavior of the R^A under right translations can be formulated as $R'^A G_A = S(g) R^A G_A S(g^{-1})$, or, without reference to any particular representation, $\theta' = g\theta g^{-1}$. Thus we obtain the transformation laws of the Maurer–Cartan form under left and right translations:

$$\theta' = \theta \quad (z' = gz), \quad \theta' = g\theta g^{-1} \quad (z' = zg) . \tag{3.19}$$

[These laws can also be derived directly from the definition (3.17). We have $\theta'(z') = \theta(z)$ under any diffeomorphism. In particular, under $z' = gz$, $\theta'(z) = \theta(g^{-1}z) = z^{-1}g d(g^{-1}z) = z^{-1} dz = \theta(z)$, and under $z' = zg$, $\theta'(z) = \theta(zg^{-1}) = gz^{-1}d(zg^{-1}) = g\theta(z)g^{-1}$.]

In our notation, the *Maurer–Cartan equation*, in terms of components, is

$$\partial_M R_N{}^A - \partial_N R_M{}^A + R_M{}^B R_N{}^C c_{BC}{}^A = 0 . \tag{3.20}$$

It is equivalent to the commutator relation (3.12).

The (anholonomic) components of the *Cartan metric* on G are

$$\gamma_{AB} = -\operatorname{tr} c_A c_B = -c_{EA}{}^F c_{FB}{}^E . \tag{3.21}$$

They satisfy

$$\gamma_{AB} = D_A{}^C D_B{}^D \gamma_{CD} . \tag{3.22}$$

Proof: Apply (3.8) in the form $G_A S = D_A{}^B S G_B$ to the case where S is itself the adjoint representation. We get immediately

$$c_{AB}{}^E D_E{}^C = D_A{}^E D_B{}^F c_{EF}{}^C$$

and (3.22) follows. The quantity

$$c_{ABC} = c_{AB}{}^D \gamma_{DC} \tag{3.23}$$

is completely skew symmetric ($c_{ABC} = -c_{AB}{}^{D}\,\mathrm{tr}\,c_D c_C = -\mathrm{tr}\,[c_A c_B]c_C$, which is easily seen to be completely skew symmetric). The equation $c_{ABC} + c_{ACB} = 0$ is in fact the infinitesimal form of (3.22). The *holonomic* components of the Cartan metric are

$$\gamma_{MN} = L_M{}^A L_N{}^B \gamma_{AB} = R_M{}^A R_N{}^B \gamma_{AB} \,. \tag{3.24}$$

The two alternative expressions are identical on account of (3.22). Since the L^A are right invariant and the R^A left invariant, the holonomic metric is both left and right invariant. In other words, the left- and right-invariant vector fields on G are Killing fields for this metric.

IV. REFERENCE SYSTEM ON $G(G/H,H)$

Suppose that coordinate systems are given on G/H and on H. Let x^i denote the coordinates of a general point $x \in G/H$ and let χ^m denote the coordinates of a general point $\chi \in H$ (the letters $i,j,\ldots$ will be used throughout as holonomic indices for G/H, and the letters $m,n,\ldots$ will be used as holonomic indices for H). With reference to a chosen section σ, the prescription (2.1) induces a coordinatization of G, whereby z is given the set of coordinates

$$z^M = (x^i,\chi^m) \,. \tag{4.1}$$

The splitting of the holonomic indices

$$M = (i,m) \tag{4.2}$$

corresponds to the local homeomorphisms between $G(G/H,H)$ and $(G/H) \otimes H$. A similar splitting of the anholonomic indices,

$$A = (\alpha,a) \tag{4.3}$$

can also be introduced. Put an extra coordinate chart V on G/H, containing πe, and an extra coordinate chart W on H, containing $(\sigma\pi e)^{-1}$. We use the letters $\alpha,\beta\ldots$ as coordinate labels for V and the letters $a,b,\ldots$ as coordinate labels for W. The prescription (2.1) now determines a coordination of $U = \{z\in G: \pi z \in V, (\sigma\pi z)^{-1} \in W\}$, whereby $g \in U$ is assigned the set of coordinates

$$g^A = (g^\alpha, g^a) \,. \tag{4.4}$$

[It is possible to be slightly more general and to use different sections for establishing the coordinatization (4.1) of G and the coordinatization (4.4) of U.]

A reference system set up according to the above procedure is adapted to the fibration in a particularly useful way. We shall call such a reference system *canonical.* Transformations relating different canonical systems consist of coordinate changes in G/H and in H, and changes of section. Of particular importance among the possible changes of the canonical reference system are changes of basis of the Lie algebra

$$G'_A = K_A{}^B G_B \,, \tag{4.5}$$

where K has the special form

$$K_A{}^B = \begin{pmatrix} \delta_\alpha^\beta & K_\alpha{}^b \\ 0 & \delta_a{}^b \end{pmatrix} . \tag{4.6}$$

We shall refer to this as a *K transformation.*

By splitting g and zg according to the precription (2.1) it is not difficult to deduce from the definition (3.2) of the right vielbein that

$$R_a{}^i = 0 \tag{4.7}$$

in a canonical system. This implies (through $RR^{-1} = 1$) that

$$R_m{}^\alpha = 0 \,. \tag{4.8}$$

Thus in a canonical reference system the sets of components of the right vielbein and of the Maurer–Cartan form have the reduced forms

$$R_A{}^M = \begin{pmatrix} R_\alpha{}^i & R_\alpha{}^m \\ 0 & R_a{}^m \end{pmatrix}, \quad R_M{}^A = \begin{pmatrix} R_i{}^\alpha & R_i{}^a \\ 0 & R_m{}^a \end{pmatrix} \tag{4.9}$$

[where, of course, the matrix $(R_i{}^\alpha)$ is the inverse of $(R_\alpha{}^i)$, and the matrix $(R_m{}^a)$ is the inverse of $(R_a{}^m)$].

Equation (4.7) means that the vector fields R_a are tangential to the fibers. They are the "vertical" left-invariant vector fields.

The commutation relations (3.17) in conjunction with (4.7) imply that $c_{ab}{}^\gamma = 0$ in a canonical system $[c_{ab}{}^\gamma R_\gamma{}^i = c_{ab}{}^C R_C{}^i = R_a{}^M \partial_M R_b{}^i - R_b{}^M \partial_M R_a{}^i = 0]$. It will be useful to display the commutation relations (3.11) in the more specific form[21]

$$\begin{aligned} [G_\alpha,G_\beta] &= c_{\alpha\beta}{}^\gamma G_\gamma + c_{\alpha\beta}{}^c G_c \,, \\ [G_a,G_\beta] &= c_{a\beta}{}^\gamma G_\gamma + c_{a\beta}{}^c G_c \,, \\ [G_a,G_b] &= c_{ab}{}^c G_c \,. \end{aligned} \tag{4.10}$$

The $c_{ab}{}^c$ are just the structure constants of the subgroup H.

The vectors R_α span at each point of the bundle space G a "horizontal" subspace of the tangent space. Considering the Lie algebra as a vector space, the left-invariant vector fields R_α (represented by the G_α) span a subspace (not in general a Lie algebra) that we shall call the *translational part* of the Lie algebra of G. These concepts are not, in general, invariant concepts: horizontal vectors are not necessarily mapped to horizontal vectors under the (right) action of the structural group H of the bundle, and the translational part of the Lie algebra can be changed by a K transformation.

Now consider the adjoint representation of G, restricted to the subgroup H. On account of $c_{ab}{}^\gamma = 0$, we have

$$D_A{}^B(h) = \begin{pmatrix} D_\alpha{}^\beta(h) & D_\alpha{}^b(h) \\ 0 & D_a{}^b(h) \end{pmatrix} . \tag{4.11}$$

The matrices $(D_a{}^b(h))$ are the matrices of the adjoint representation of H and the matrices $(D_\alpha{}^\beta(h))$ provide a special matrix representation of H with the dimensionality of G/H. If $D_\alpha{}^b(h) = 0$ for all $h \in H$ or if all the $D_\alpha{}^b(h)$ can be transformed away by a K transformation, the space G/H is called *reductive*. In that case, there is a canonical system in which[21]

$$c_{\alpha c}{}^b = 0 \,. \tag{4.12}$$

The commutator $[R_\alpha,R_c]$ is then a linear combination of the R_β; the horizontal spaces are preserved by the (right) action of the structural group H and we have a "connection" in the Ehresmann sense on the fiber bundle $G(G/H,H)$. The components R^a of the Maurer–Cartan form of G are the components of the connection one-form of this Ehresmann connection. Thus the components R^a comprise a connection one-form on $G(G/H,H)$ if and only if G/H is reductive (see, for example, Kobayashi and Nomizu,[1] p. 103). In our scheme, we shall not insist that G/H be reductive. Indeed, the case when G/H is *not* reductive is especially interesting.

V. INFINITESIMAL GENERATORS AND COVARIANT DERIVATIVES OF FIELDS ON *G/H*

Let Ψ be a set of scalar fields on G, satisfying a fiber condition (2.3). We define the operators M_A and $\mathring{Q}_A$, which act on the pullback $\Psi = \sigma^*\Psi$ as follows:

$$M_A\psi = \sigma^*(L_A\Psi)\,, \tag{5.1}$$

$$\mathring{Q}_A\psi = \sigma^*(R_A\Psi)\,. \tag{5.2}$$

An infinitesimal left translation $z'^M = z^M - a^A L_A{}^M$ induces a diffeomorphism on G/H given by

$$x^i = x^i - a^A L_A{}^i\,. \tag{5.3}$$

[It is instructive to note that the fact that the quantities $L_A{}^i$ are functions of x only, $L_A{}^i = L_A{}^i(x)$ independent of χ, can be inferred from the commutation relations (3.14) together with (4.7); we have

$$R_a{}^m\,\partial_m L_A{}^i = R_a{}^M\,\partial_M L_A{}^i - L_A{}^M\,\partial_M R_a{}^i$$
$$= [R_a,L_A]^i = 0\,.$$

So $\partial_m L_A{}^i = 0$.] Under the diffeomorphism (5.3), the transformation law of ψ is

$$\delta\psi = a^A M_A\psi\,. \tag{5.4}$$

This is the infinitesimal form of the nonlinear transformation law (2.7). The M_A are the generators of infinitesimal left translations, for the nonlinearity transforming ψ. Interpreting G/H as space-time, the transformation laws (5.3) and (5.4) give the action of a space-time symmetry, or combination of a space-time symmetry and an internal symmetry, on the points of space-time and on physical fields [for example, if G is SO(4,2) and H the 11-parameter subgroup corresponding to the subgroup of the conformal group that consists of Lorentz rotations, dilatations, and special conformal transformations, we will get the action of the conformal group on Minkowski space-time and on physical fields[30]].

Since the fields $L_A\Psi$ satisfy a fiber condition $(L_A\Psi)' = \overline{S}(h)L_A\Psi$ $(z' = zh)$ (a consequence of the right invariance of the L_A), the action of M_A on $M_B\psi$ is well defined. We have $M_A M_B\psi = \sigma^*(L_A L_B\Psi)$. Therefore the operators M_A satisfy the same commutation relations as the L_A:

$$[M_A,M_B] = -c_{AB}{}^C M_C\,. \tag{5.5}$$

Similarly, since the fields R_A satisfy a fiber condition

$$(R_A\Psi)' = D_A{}^B(h)\overline{S}(h)R_A\Psi \quad (z' = zh)$$

[a consequence of the transformation law of the R_A under the right action of H, given by (3.16)], we can deduce that $\mathring{Q}_A\mathring{Q}_B\Psi = \sigma^*(R_A R_B\Psi)$ and therefore that the operators $\mathring{Q}_A$ satisfy the same commutation relations as the R_A:

$$[\mathring{Q}_A,\mathring{Q}_B] = c_{AB}{}^C\mathring{Q}_C\,. \tag{5.6}$$

In a similar manner, we can deduce also that

$$[M_A,\mathring{Q}_B] = 0\,. \tag{5.7}$$

The fiber condition on $R_A\Psi$ leads immediately, as a consequence of the considerations of Sec. II, to the following transformation law of the fields $\mathring{Q}_A$ under a left translation:

$$(\mathring{Q}_A\psi)'(x') = D_A{}^B(h(g,x))\overline{S}(h(g,x))(\mathring{Q}_B\psi)(x)$$
$$(z' = gz)\,. \tag{5.8}$$

For reasons that will become apparent, we shall call $\mathring{Q}_A\psi$ a *covariant derivative* of ψ. [For the present, observe only that $\mathring{Q}_A\psi$ is constructed from the field ψ on *G/H and its derivatives* $\partial_i\psi$ and that $\mathring{Q}_A\psi$ transforms under the group G of left translations homogeneously, in spite of the fact that the transformation matrix $\overline{S}(h(g,x))$ is x dependent. These properties are characteristic of a covariant derivative.]

Finally, note the relation

$$M_A = D_A{}^B(\sigma)\mathring{Q}_B\,, \tag{5.9}$$

which is a direct consequence of the definitions of the operators M_A and $\mathring{Q}_A$.

VI. A CONNECTION ON *G/H*

We define the *connection* on G/H, associated with the group G of left translations, to be the pullback of the Maurer–Cartan form,

$$\mathring{\Gamma} = \sigma^*\theta\,. \tag{6.1}$$

Since $\pi\sigma = 1$, the components of $\sigma(x)$ have the form $\sigma^M(x) = (x^i,\sigma^m(x))$. The section is determined by the functions $\sigma^m(x)$. In terms of components, the definition (6.1) has the more explicit form

$$\mathring{\Gamma}_i{}^A(x) = \sigma^M{}_{,i}R_M{}^A(\sigma) = R_i{}^A(\sigma) + \sigma^m{}_{,i}R_m{}^A(\sigma)\,. \tag{6.2}$$

In particular, the components of the translational part of the connection are

$$\mathring{e}_i{}^\alpha = \mathring{\Gamma}_i{}^\alpha = R_i{}^\alpha(\sigma)\,. \tag{6.3}$$

The inverse $(\mathring{e}_\alpha{}^i)$ of the matrix $(\mathring{e}_i{}^\alpha)$ provides a set of components of a *vielbein* on G/H. Since we are interpreting G/H as a space-time, they are the components of a *tetrad*.

Let us define the matrices

$$\mathring{\Gamma}_i = \mathring{\Gamma}_i{}^A G_A = \mathring{e}_i{}^\alpha G_\alpha + \mathring{\Gamma}_i{}^a G_a\,, \tag{6.4}$$

where the G_A are the generators of any matrix representation S of G. In terms of the matrices (6.4), the definition (6.2) can be written as $\mathring{\Gamma}_i = \sigma^M{}_{,i}R_M{}^A(\sigma)G_A$. But from (3.7) we have $R_M{}^A(\sigma)G_A = S^{-1}(\sigma)S_{,M}(\sigma)$, so

$$\mathring{\Gamma}_i = \sigma^M{}_{,i}S^{-1}(\sigma)S_{,M}(\sigma) = S^{-1}(\sigma)\partial_i S(\sigma)\,.$$

So we can write, symbolically without reference to a particular representation,

$$\mathring{\Gamma}_i = \sigma^{-1}\,\partial_i\sigma\,. \tag{6.5}$$

Since the Maurer–Cartan form is left invariant, and since the section is not changed when a left translation is applied, the connection $\mathring{\Gamma}$ is left invariant,

$$\mathring{\Gamma}' = \mathring{\Gamma} \quad (z' = gz)\,. \tag{6.6}$$

Therefore

$$\mathring{\Gamma}_i{}'(x') = \mathring{\Gamma}_i(x') = (\sigma(x'))^{-1}\frac{\partial\sigma(x)}{\partial x'^i}$$

$$= \frac{\partial x^j}{\partial x'^i}(h\sigma^{-1}g^{-1})\partial_j(g\sigma h^{-1})$$

$$= \frac{\partial x^j}{\partial x'^i}(h(\sigma^{-1}\partial_j\sigma)h^{-1} + h\,\partial_j h^{-1}).$$

The transformation law of the components of $\mathring{\Gamma}$ under a left translation, under which they remain invariant, is therefore

$$\mathring{\Gamma}_i{}'(x') = \mathring{\Gamma}_i(x) = \frac{\partial x^j}{\partial x'^i}(h\mathring{\Gamma}_j h^{-1} - \partial_j h\cdot h^{-1}), \quad (6.7)$$

where

$$h = h(g,x) \quad (6.8)$$

is given by (2.2).

An alternative form of this transformation law is

$$\mathring{\Gamma}'_i{}^A(x') = \mathring{\Gamma}_i{}^A(x')$$

$$= \frac{\partial x^j}{\partial x'^i}(\mathring{\Gamma}_j{}^B(x)D_B{}^A(h^{-1})$$

$$+ \partial_j(h^{-1})^m\cdot R_m{}^A(h^{-1})). \quad (6.9)$$

That this is equivalent to (6.7) can be verified as follows. Multiply (6.9) by G_A and apply (3.8) to the first term. For the second term, we employ (3.7) in the form $R_M{}^A\cdot G_A = S^{-1}S_{\cdot M}$. This implies

$$\partial_j(h^{-1})^m R_m{}^A(h^{-1})G_A$$

$$= \partial_j(h^{-1})^m S(h)\partial_m S(h^{-1}) = S(h)\,\partial_j S(h^{-1}).$$

Equation (6.9) has now become

$$\mathring{\Gamma}_i{}'(x') = \mathring{\Gamma}_i(x')$$

$$= \frac{\partial x^j}{\partial x'^i}(S(h)\mathring{\Gamma}_j(x)S(h^{-1}) + S(h)\partial_j S(h^{-1})),$$

which is just (6.7) evaluated for a particular representation S of G. A direct derivation of (6.9) is rather more complicated. We have

$$\mathring{\Gamma}_i{}'^A(x') = \mathring{\Gamma}_i{}^A(x') = \frac{\partial\sigma^M(x')}{\partial x'^i}R_M{}^A(\sigma(x'))$$

$$= \frac{\partial x^j}{\partial x'^i}\frac{\partial(g\sigma h^{-1})^M}{\partial x^j}R_M{}^A(\sigma(x'))$$

and

$$\frac{\partial(g\sigma h^{-1})^M}{\partial x^j}R_M{}^A(\sigma(x'))$$

$$= \frac{\partial(\sigma h^{-1})^N}{\partial x^j}\frac{\partial(g\sigma h^{-1})^M}{\partial(\sigma h^{-1})^N}R_M{}^A(\sigma(x'))$$

$$= \left[\frac{\partial\sigma^P}{\partial x^j}\frac{\partial(\sigma h^{-1})^N}{\partial\sigma^P} + \frac{\partial(h^{-1})^m}{\partial x^j}\frac{\partial(\sigma h^{-1})^N}{\partial(h^{-1})^m}\right]$$

$$\times[R^{-1}(\sigma h^{-1})R(g\sigma h^{-1})]_N{}^M R_M{}^A(\sigma(x')).$$

The transformation law (6.9) implies the transformation law

$$\mathring{e}'_i{}^\alpha(x') = \mathring{e}_i{}^\alpha(x') = \frac{\partial x^j}{\partial x'^i}\mathring{e}_j{}^\beta D_\beta{}^\alpha(h^{-1}) \quad (6.10)$$

for the tetrad components, under the action of a left translation.

The Maurer–Cartan equation implies a zero curvature for the connection on G/H,

$$\partial_i\mathring{\Gamma}_j{}^A - \partial_j\mathring{\Gamma}_i{}^A + \mathring{\Gamma}_i{}^B\mathring{\Gamma}_j{}^C c_{BC}{}^A = 0. \quad (6.11)$$

Proof: Let $R^{\ A}_{M\cdot N}(\sigma)$ denote $\partial_N R^{\ A}_M$ evaluated at $\sigma(x)$. Then

$$\partial_i\mathring{\Gamma}_j^A - \partial_j\mathring{\Gamma}_i^A = \partial_i(\sigma^M{}_{\cdot j}R^{\ A}_M(\sigma)) - \partial_j(\sigma^M{}_{\cdot i}R^{\ A}_M(\sigma))$$

$$= \sigma^M{}_{\cdot j}\sigma^N{}_{\cdot i}(R^{\ A}_{M\cdot N}(\sigma) - R^{\ A}_{N\cdot M}(\sigma))$$

$$= \sigma^M{}_{\cdot i}\sigma^N{}_{\cdot j}R^{\ B}_M(\sigma)R^{\ C}_N(\sigma)c_{BC}{}^A$$

$$= \mathring{\Gamma}_j{}^B\mathring{\Gamma}_i{}^C c_{BC}{}^A.$$

We are now in a position to obtain a more explicit expression for the covariant derivative $\mathring{Q}_A\psi$ of a field on G/H. We have

$$\mathring{Q}_A\psi = \sigma^*(R_A\Psi) = R_A{}^M(\sigma)\Psi_{\cdot M}(\sigma), \quad (6.12)$$

where $\Psi_{\cdot M}(\sigma)$ denotes $\partial_M\Psi$ evaluated at the point $\sigma(x)$, so that

$$\partial_i\psi = \partial_i\Psi(\sigma) = (\partial_i\sigma^M)\Psi_{\cdot M}(\sigma)$$

$$= \Psi_{\cdot i}(\sigma) + (\partial_i\sigma^m)\Psi_{\cdot m}(\sigma). \quad (6.13)$$

Differentiate the fiber condition $\Psi(z) = \overline{S}(h)\Psi(zh)$ with respect to h^a and set $h = e$. We got $0 = \overline{G}_a\Psi + R_a\Psi = \overline{G}_a\Psi + R_a{}^M\,\partial_M\Psi$ (where the $\overline{G}_a$ are the generators of the matrix representation $\overline{S}$ of H). Hence

$$\Psi_{\cdot m}(\sigma) = -R_m{}^a(\sigma)\overline{G}_a\psi. \quad (6.14)$$

By substituting (6.13) and (6.14) into (6.12), we find

$$\mathring{Q}_\alpha\psi = \mathring{D}_\alpha\psi = \mathring{e}_\alpha{}^i\mathring{D}_i\psi, \quad \mathring{D}_i\psi = \partial_i\psi + \mathring{\Gamma}_i{}^a\overline{G}_a\psi, \quad (6.15)$$

and

$$\mathring{Q}_a\psi = -\overline{G}_a\psi. \quad (6.16)$$

Observe that, although $\mathring{D}_i$ *looks* like a convariant derivative associated with the group H, in general it is not. This phenomenon was encountered already in our earlier work.[21] The present fiber bundle description gives a much clearer geometrical insight into what is happening. The $\mathring{Q}_\alpha\psi$ and $\mathring{Q}_a\psi$ are two pieces of a single geometrical entity, which transforms according to the transformation law (5.8). *Only when G/H is reductive* do these two pieces transform independently and in that case $D_i\psi$ transforms like a true covariant derivative for the subgroup H of the group G of left translations.

VII. METRICS INDUCED ON *G/H*

In certain circumstances the diffeomorphisms induced on G/H by left translations on G are closely related to naturally arising metrical properties of G/H.

The most straightforward case arises when G is semisimple, so that the Cartan metric γ_{AB} has an inverse γ^{AB}, and *in addition* the submatrix $\gamma^{\alpha\beta}$ is nonsingular (we shall denote its inverse by $\eta_{\alpha\beta}$). We can in that case define a nonsingular rank 2 tensor field on G/H,

$$g^{ij} = \gamma^{AB}L_A{}^iL_B{}^j = \gamma^{AB}R_A{}^iR_B{}^j = \gamma^{\alpha\beta}R_\alpha{}^iR_\beta{}^j = \gamma^{\alpha\beta}\mathring{e}_\alpha{}^i\mathring{e}_\beta{}^j. \quad (7.1)$$

Since the quantities $L_A{}^i$ depend only on x, the g_{ij} are indeed uniquely defined on G/H. The fact that the vectors R_A are left invariant ensures that the g^{ij} are invariant under the diffeomorphisms induced on G/H by the left translations. These diffeomorphisms are therefore *isometries* for the metric on G/H with components $g_{ij} = \mathring{e}_i{}^\alpha \mathring{e}_j{}^\beta \eta_{\alpha\beta}$. The $L_A{}^i$ are the components of a set of Killing vectors on G/H. The transformation law of the metric of G/H under left translations (under which the components remain invariant) is

$$g_{ij}(x') = g'_{ij}(x') = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} g_{kl}(x) . \tag{7.2}$$

An example of an induced metric of this kind is the metric of de Sitter space G/H, where G is SO(4,1).

If the conditions on G that lead to the above construction of a metric on G/H are not satisfied, there may nevertheless exist a constant nonsingular matrix $\eta_{\alpha\beta}$ such that[21]

$$D_\alpha{}^\gamma(h) D_\beta{}^\delta(h) \eta_{\gamma\delta} = \rho(h) \eta_{\alpha\beta} \tag{7.3}$$

for every $h \in H$ (ρ is therefore a one-dimensional representation of H). In that case we can regard

$$g_{ij} = \mathring{e}_i{}^\alpha \mathring{e}_j{}^\beta \eta_{\alpha\beta} \tag{7.4}$$

as the components of a metric on G/H. It is left invariant because the tetrad is; under left translations $g'_{ij} = g_{ij}$. But the diffeomorphisms on G/H induced by left translations are not, *in general*, isometries because the transformation law of g_{ij} under left translation is not the usual tensor transformation law (7.2). The transformation law (6.10) of the tetrad gives

$$g'_{ij}(x') = g_{ij}(x') = \rho^{-1}(h(g,x)) \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} g_{kl}(x) . \tag{7.5}$$

In the simplest case, $\rho = 1$ and the left translations induce isometries. An example of this is the action of the Poincaré group on Minkowski space. More generally, the left translations induce *conformal* mappings on G/H. Examples are the action of the conformal group SO(4,2) on Minkowski space and the action of the conformal group SO(4,2) on de Sitter space. These two cases are related by a K transformation.

Finally, there are cases in which a metric cannot be induced on G/H by the above method because no $\eta_{\alpha\beta}$ with the required property exists. An example of this is the action of the affine group on space-time.

VIII. GAUGING THE LEFT TRANSLATIONS

Up to this point, we have dealt only with the formalism associated with the *ungauged* group G of left translations. We shall now introduce those diffeomorphisms on G that can be regarded as the gauged generalizations of left translations.

Observe first that the (ungauged) left translations are just those diffeomorphisms $z \to z'$ on G that satisfy

$$z' g_0 = (z g_0)', \quad \text{for all } g_0 \in G . \tag{8.1}$$

We now define a *gauge transformation* to be a bundle automorphism, that is, a diffeomorphism $z \to z'$ on G that satisfies

$$z' h = (zh)', \quad \text{for all } h \in H . \tag{8.2}$$

(This concept of gauge transformation appears in the work of Atiyah, Hitchin, and Singer,[31] but we have abandoned the requirement that the action induced on base space shall be trivial.)

It then follows that the gauge transformations are just those diffeomorphisms on G of the form

$$z' = g(x) z , \tag{8.3}$$

where $x = \pi z$ and $g(x)$ is a G-valued function on G/H. [As proof, define $g(z) = z' z^{-1}$. Then (8.3) implies $g(z) = g(zh)$, for all $h \in H$. So $g(z)$ is constant on each fiber. We have a G-valued function $g(x) = g(z)$ on G/H.]

Observe that the form (8.3) of a gauge transformation is in agreement with the elementary concept of gauging a symmetry group, due to Yang and Mills; the group action is generalized by allowing the group elements to be space-time dependent.

It is important to note that not every G-valued field $g(x)$ on G/H defines a gauge transformation through (8.3). In general, a mapping on G of the form (8.3) will not even be one-to-one.

The geometrical meaning of (8.2) is that the gauge transformations are those diffeomorphisms on G that map fibers to fibers and preserve the action of the structural group H of $G(G/H,H)$. That is, two points on the same fiber related to each other by right multiplication by $h \in H$ will have two images related in the same way.

The prescription (2.2) generalizes immediately to the case of a gauge transformation. Simply replace g by $g(x)$ in (2.2). Figure 1 now illustrates this more general situation. In other words, a gauge transformation induces a diffeomorphism $x \to x'$ on G/H and specifies a unique H-valued field $h(x)$ on G/H, through the prescription[32]

$$h(x) = (\sigma(x'))^{-1} g(x) \sigma(x) . \tag{8.4}$$

Conversely, any diffeomorphism $x \to x'$ on G/H together with any H-valued field $h(x)$ on G/H determine a unique gauge transformation $z' = g(x) z$, through the prescription

$$g(x) = \sigma(x') h(x) (\sigma(x))^{-1} . \tag{8.5}$$

In Sec. II we obtained the transformation law of a field ψ on G/H, defined as the pullback of a scalar field Ψ on G satisfying a fiber condition. This transformation law generalizes immediately: the transformation law of ψ under a gauge transformation is simply

$$\psi'(x') = \overline{S}(h(x)) \psi(x) . \tag{8.6}$$

Consider now the *infinitesimal* gauge transformations. We have already seen that an infinitesimal (ungauged) left translation $z'^M = z^M - \Lambda^M$ is generated by an infinitesimal vector Λ that is *right invariant*,

$$[R_A, \Lambda] = 0 , \tag{8.7}$$

or, equivalently,

$$\Lambda = a^A L_A \quad (a^A \text{ const}) . \tag{8.8}$$

An infinitesimal gauge transformation is generated by an infinitesimal vector Λ that is *invariant under the right action of the subgroup H*,

$$[R_a, \Lambda] = 0 , \tag{8.9}$$

or, equivalently,

$$\Lambda = a^A L_A \quad (a^A = a^A(x)) . \tag{8.10}$$

The effect of an infinitesimal gauge transformation on the points of G/H and on the fields ψ on G/H is obtained as follows. Define $\lambda^M = \Lambda^M(\sigma)$. Then the diffeomorphism $x \to x'$ induced on G/H by an infinitesimal gauge transformation $z'^M = z^M - \Lambda^M = z^M - a^A(x)L_A{}^M$ is $x'^i = x^i - \lambda^i$ (note that Λ^i is dependent only on x anyway, so in fact $\lambda^i = \Lambda^i$). The transformation law of a scalar field Ψ on G is $\delta\Psi = \Lambda^M \partial_M \Psi$ so the infinitesimal form of (8.6) is $\delta\psi = \sigma^*(\Lambda^M \partial_M \Psi) = a^A M_A \psi$. Alternatively, regarding the λ^M as the (space-time-dependent) parameters rather than the a^A, $\delta\psi = \lambda^M \Psi_{.M}(\sigma)$. Employing formulas (6.13) and (6.14) we find

$$\delta\psi = \lambda^i \partial_i \psi - \epsilon^a \overline{G}_a \psi, \tag{8.11}$$

where

$$\epsilon^a = (\lambda^M - \lambda^i \sigma^M{}_{,i}) R_M{}^a(\sigma) = (\lambda^m - \lambda^i \sigma^m{}_{,i}) R_m{}^a(\sigma). \tag{8.12}$$

The form (8.11) of the transformation law of ψ emphasizes the fact that a gauge transformation consists of a general diffeomorphism on the space-time G/H together with a space-time-dependent element of H. The geometrical explanation of the peculiar form of the parameters ϵ^a is given by Fig. 2; the vector λ can be built up from a component tangential to the section and a vertical component ϵ. The ϵ^a are the anholonomic components of ϵ.

IX. DIFFERENTIAL GEOMETRY OF A VIELBEIN

As a preliminary to the construction of a connection and curvature associated with our gauge transformations, we shall consider the differential geometry of a vielbein field on a differentiable manifold.

Denote the coordinates of the general point of a manifold by z^M (M,N holonomic indices; $A,B,\ldots$ anholonomic indices). Let E be the matrix of components $E_A{}^M$ of a vielbein and denote the matrix elements of E^{-1} by $E_M{}^A$. The vielbein vector fields are $E_A = E_A{}^M \partial_M$ and the dually associated one-form fields are $E^A = dz^M E_M{}^A$. The components $\Omega_{AB}{}^C$ of the "object of anholonimity" are defined by

$$[E_A, E_B] = \Omega_{AB}{}^C E_C. \tag{9.1}$$

Using the vielbein components to convert anholonomic indices to holonomic indices and vice versa, in the usual way, we have

$$\partial_M E_N{}^A - \partial_N E_M{}^A = -\Omega_{MN}{}^A. \tag{9.2}$$

Under the action of an infinitesimal diffeomorphism $z'^M = z^M - \Lambda^M$, the transformation law of the vielbein is

$$\delta E_A = [\Lambda, E_A], \tag{9.3}$$

which leads to

$$\delta E_M{}^A = \mathscr{D}_M \Lambda^A \tag{9.4}$$

where

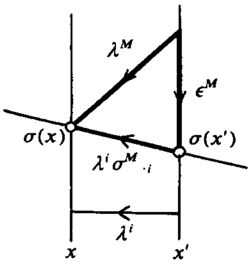

FIG. 2. The geometrical explanation of the form of the parameters ϵ^a.

$$\mathscr{D}_M \Lambda^A = \partial_M \Lambda^A + \Lambda^B \Omega_{MB}{}^A. \tag{9.5}$$

Observe incidentally that the $\Omega_{MB}{}^A$ are the anholonomic components of the linear connection whose holonomic components are

$$\Gamma_{MN}{}^P = (\partial_N E_M{}^A) E_A{}^P = -E_M{}^A \partial_N E_A{}^P. \tag{9.6}$$

The operator $\mathscr{D}_M$ is then a covariant derivative operator. We have, for example, for a contravariant vector A^P,

$$\mathscr{D}_M A^P = \partial_M A^P + \Lambda^N \Gamma_{MN}{}^P. \tag{9.7}$$

The linear connection (9.5) has vanishing curvature but nonvanishing torsion $\Omega_{MN}{}^P$. With respect to this connection the manifold is a "space of distant parallelism."

When the manifold is the manifold of a Lie group G, the fields $G_{AB}{}^C = \Omega_{AB}{}^C - c_{AB}{}^C$ are of particular importance. We have

$$[E_A, E_B] - c_{AB}{}^C E_C = G_{AB}{}^C E_C \tag{9.8}$$

or equivalently,

$$\partial_M E_N{}^A - \partial_N E_M{}^A + E_M{}^B E_N{}^C c_{BC}{}^A = -G_{MN}{}^A. \tag{9.9}$$

Note that $G_{MM}{}^A$ vanishes for the right vielbein. The transformation law (9.4) can be rewritten in the form

$$\delta E_M{}^A = \Lambda^N G_{MN}{}^A + \nabla_M \Lambda^A, \tag{9.10}$$

where

$$\nabla_M \Lambda^A = \partial_M \Lambda^A - \Lambda^B E_M{}^C c_{BC}{}^A. \tag{9.11}$$

X. CONNECTION AND CURVATURE FOR THE GAUGE TRANSFORMATIONS

We shall now show how the formalism of our earlier work[21] on the gauging of a group G of space-time and internal symmetries arises as a particular case of the fiber bundle geometry.[33]

Define a *connection* on $G(G/H,H)$ to be the set of one-forms E^A dual to a vielbein E_A on the group manifold G, satisfying the following two conditions.

(1) The vielbein satisfies the same fiber condition as the right vielbein, namely

$$E'_A = D_A{}^B(h) E_B \quad (z' = zh), \tag{10.1}$$

for every $h \in H$ [see (3.16)].

(2) The vertical vectors of the vielbein are the same as those of the right vielbein ($E_a{}^M = R_a{}^M$). Thus in a canonical reference system the components of the specialized vielbein and its dually associated connection are

$$E_A{}^M = \begin{pmatrix} E_\alpha{}^i & E_\alpha{}^m \\ 0 & R_a{}^m \end{pmatrix}, \quad E_M{}^A = \begin{pmatrix} E_i{}^\alpha & E_i{}^a \\ 0 & R_m{}^a \end{pmatrix}. \tag{10.2}$$

The above two conditions are gauge invariant. The gauge invariance of (1) follows simply from the fact that gauge transformations, by definition, commute with the right action of H. Condition (2), $E_a = R_a$, transforms to $E'_a = R'_a$ (under a gauge transformation $z \to z'$). But by (8.9), the R_a are invariant, $R'_a = R_a$.

We call the set of two-forms whose components are $G_{MN}{}^A$ the *curvature* associated with the connection E^A. The infinitesimal form of the transformation law (10.1) is $[R_a, E_B] = c_{aB}{}^C E_C$. Since $R_a = E_a$, this implies that

$G_{\alpha B}{}^C = 0$. Therefore $G_{mN}{}^C = E^A_m E^B_N G_{AB}{}^C = 0$. Thus the only nonvanishing components of the curvature are $G_{ij}{}^A$.

We define the connection on G/H to be the pullback of the connection on G,

$$\Gamma^A = \sigma^* E^A . \tag{10.3}$$

Its components are

$$\Gamma_i{}^A = \sigma^M{}_{.i} E_M{}^A(\sigma) . \tag{10.4}$$

In particular,

$$e_i{}^\alpha = \Gamma_i{}^\alpha = E_i{}^\alpha(\sigma) \tag{10.5}$$

defines a tetrad on the space-time G/H. Because of the conditions (1) and (2), the $E_M{}^A$ are completely determined by the $\Gamma_i{}^A$.

We define the curvature on G/H to be the pullback of the curvature on G. Its components are

$$\sigma^M{}_{.i}\sigma^N{}_{.j} G_{MN}{}^A(\sigma) = G_{ij}{}^A(\sigma) . \tag{10.6}$$

Observe that

$$\partial_i \Gamma_j{}^A = \partial_i(\sigma^M{}_{.j} E_M{}^A(\sigma))$$
$$= \sigma^M{}_{.ji} E_M{}^A(\sigma) + \sigma^M{}_{.j}\sigma^N{}_{.i} E^A{}_{M\cdot N}(\sigma) .$$

So

$$\partial_i\Gamma_j{}^A - \partial_j\Gamma_i{}^A = \sigma^M{}_{.j}\sigma^N{}_{.i}(E^A_{M\cdot N}(\sigma) - E^A_{N\cdot M}(\sigma))$$
$$= \sigma^M{}_{.j}\sigma^N{}_{.i}(G_{MN}{}^A(\sigma) - E^B_M(\sigma)E^C_N(\sigma)c_{BC}{}^A) .$$

Therefore the components of the curvature on G/H are given by

$$\partial_i\Gamma_j{}^A - \partial_j\Gamma_i{}^A + \Gamma_i{}^B\Gamma_j{}^C c_{BC}{}^A = -G_{ij}{}^A \tag{10.7}$$

[it is convenient to drop the argument, writing simply $G_{ij}{}^A$ to mean $G_{ij}{}^A(\sigma)$].

Denote the components of the pullback of the object of anholonimity by $F_{AB}{}^C$. The quantities $F_{ij}{}^\alpha$ and $F_{ij}{}^a$ defined by

$$F_{ij}{}^A = e_i{}^\beta e_j{}^\gamma F_{\beta\gamma}{}^A \tag{10.8}$$

are the components of the H torsion and the H curvature.[21]

The infinitesimal vector field Λ^M that generates an infinitesimal gauge transformation has anholonomic components $\Lambda^A = \Lambda^M E_M{}^A$. Since this vector field is invariant under the right action of H, it is completely determined by either $\lambda^M = \Lambda^M(\sigma)$ or by $\lambda^A = \Lambda^A(\sigma)$, either of which can be regarded as the set of (space-time-dependent) parameters of the infinitesimal gauge transformation. The transformation law for the components of the connection on G/H under an infinitesimal gauge transformation are now easily found. We have $\delta\Gamma_i{}^A = \sigma^M{}_{.i}\delta E^{MA} = \sigma^M{}_{.i}\Lambda^A{}_{.M}(\sigma) - \lambda^B\Gamma_i{}^c F_{BC}{}^A$ (from 9.4). Therefore

$$\delta\Gamma_i{}^A = \mathscr{D}_i\lambda^A , \tag{10.9}$$

where

$$\mathscr{D}_i\lambda^A = \partial_i\lambda^A - \lambda^B\Gamma_i{}^C F_{BC}{}^A . \tag{10.10}$$

An alternative form of (10.9) is

$$\delta\Gamma_i{}^A = \nabla_i\lambda^A - \lambda^j G_{ji}{}^A, \tag{10.11}$$

where

$$\nabla_i\lambda^A = \partial_i\lambda^A - \lambda^B\Gamma_i{}^C c_{BC}{}^A \tag{10.12}$$

(note that $\lambda^i = \lambda^\alpha e_\alpha{}^i$). An alternative set of parameters for an infinitesimal gauge transformation consists of the λ^i and the components ϵ^a of the vertical vector introduced in (8.12) and Fig. 2. We have

$$\epsilon^A = (\lambda^M - \lambda^i\sigma^M{}_{.i})E_M{}^A(\sigma) = \lambda^A - \lambda^i\Gamma_i{}^A \tag{10.13}$$

(which satisfies $\epsilon^\alpha = 0$). This relation appeared already in our earlier work. The fiber bundle formalism gives it a very clear geometrical meaning. In terms of these parameters,

$$\delta\Gamma_i{}^A = \lambda^j\partial_j\Gamma_i{}^A + (\partial_i\lambda^j)\Gamma_j{}^A + \nabla_i\epsilon^A . \tag{10.14}$$

This shows that $\Gamma_i{}^A$ transforms like a one-form under the diffeomorphism on G/H induced by a gauge transformation, and "like a Yang–Mills potential for an internal symmetry group G" under the action of the H-valued field on G/H associated with the gauge transformation. The transformation law under a *finite* gauge transformation is therefore a straightforward generalization of the transformation law given in Sec. VI for the "trivial" connection $\mathring{\Gamma}_i$ under left translations. A finite gauge transformation consists of a general diffeomorphism $x \to x'$ on G/H and an H-valued field $h = h(x)$ on G/H, according to (8.4). The transformation law of the components of the connection on G/H is

$$\Gamma'_i{}^A(x') = \frac{\partial x^j}{\partial x'^i}(\Gamma_j{}^B(x)D_B{}^A(h^{-1}) + \partial_j(h^{-1})^m R_m{}^A(h^{-1})) , \tag{10.15}$$

or equivalently, for $\Gamma_i = \Gamma_i{}^A G_A = e_i{}^\alpha G_\alpha + \Gamma_i{}^a G_a$,

$$\Gamma'_i(x') = \frac{\partial x^j}{\partial x'^i}(h\Gamma_i(x)h^{-1} - \partial_i h\cdot h^{-1}) . \tag{10.16}$$

In particular, the tetrad transforms according to

$$e'_i{}^\alpha(x') = \frac{\partial x^j}{\partial x'^i} e_j{}^\beta(x) D_\beta{}^\alpha(h^{-1}) . \tag{10.17}$$

The transformation law of the curvature $G_{ij}{}^A$ on G/H is

$$G'_{ij}{}^A(x') = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} G_{kl}{}^B D_B{}^A(h^{-1}) \tag{10.18}$$

or equivalently, for $G_{ij} = G_{ij}{}^A G_A = G_{ij}{}^\alpha G_\alpha + G_{ij}{}^a G_a$,

$$G'_{ij}(x') = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} hG_{ij}(x)h^{-1} . \tag{10.19}$$

It can be shown that *if G/H is reductive*, the $F_{ij}{}^A$ transform like the $G_{ij}{}^A$. Otherwise, they have a complicated inhomogeneous transformation law. If the translational part of the Lie algebra is Abelian, then $F_{ij} = G_{ij}$.

In cases when a nonsingular matrix $\eta_{\alpha\beta}$ satisfying the conditions of Sec. VII exists, if follows from (10.17) that the transformation law of the G/H metric $g_{ij} = e_i{}^\alpha e_j{}^\beta\eta_{\alpha\beta}$ is

$$g'_{ij}(x') = \rho(h^{-1})\frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} g_{kl}(x) . \tag{10.20}$$

Thus, in general, in such cases gauge transformations induce Weyl (scale) transformations on the space-time metric, as well as diffeomorphisms.

XI. THE GENERALIZED COVARIANT DERIVATIVE

The covariant derivative operator associated with (ungauged) left translations introduced in Sec. V is readily generalized to a derivative operator covariant under gauge transformations. We simply define

$$Q_A \psi = \sigma^* (E_A \Psi) . \tag{11.1}$$

Since the E_A satisfy a fiber condition, the action of Q_A on $Q_B \psi$ is well defined, and in fact $Q_A Q_B \psi = \sigma^*(E_A E_B \Psi)$. Then (9.1) and (10.8) imply

$$[Q_A, Q_B] = F_{AB}{}^C Q_C , \tag{11.2}$$

or, more explicity,

$$\begin{aligned} [Q_\alpha, Q_\beta] &= F_{\alpha\beta}{}^\gamma Q_\gamma + F_{\alpha\beta}{}^c Q_c , \\ [Q_a, Q_\beta] &= c_{a\beta}{}^\gamma Q_\gamma + c_{a\beta}{}^c Q_c , \\ [Q_a, Q_b] &= c_{ab}{}^c Q_c . \end{aligned} \tag{11.3}$$

In the particular case of Poincaré gauge theory, this "generalized Lie algebra" is already well known.[10] The arguments that led to (6.15) and (6.16) now provide the following explicit expressions for the generalized operators Q_A:

$$\begin{aligned} &Q_\alpha \psi = D_\alpha \psi = e_\alpha{}^i D_i \psi, \quad D_i \psi = \partial_i \psi + \Gamma_i{}^a \overline{G}_a \psi , \\ &Q_a \psi = -\overline{G}_a \psi . \end{aligned} \tag{11.4}$$

The covariant transformation law of $Q_A \psi$ under a gauge transformation is a straightforward generalization of (5.8),

$$(Q_A \psi)'(x') = D_A{}^B(h) \overline{S}(h) (Q_B \psi)(x) . \tag{11.5}$$

The transformation law (8.11) of ψ under an infinitesimal gauge transformation can be reexpressed in terms of the parameters λ^A. We have $\delta\psi = \sigma^* \delta\Psi = \sigma^*(\Lambda^A E_A \Psi) = \lambda^A \sigma^*(E_A \Psi)$. Therefore

$$\delta\psi = \lambda^A Q_A \psi = \lambda^\alpha D_\alpha \psi - \lambda^a \overline{G}_a \psi . \tag{11.6}$$

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[32]It is to be understood that, when σ is a collection of local sections σ_α with supports $U_\alpha \in G/H$, U_α a covering of G/H, (8.4) becomes a prescription for a collection of H-valued functions $h_{\alpha\beta}$ defined by $h_{\alpha\beta}(x) = \sigma_\beta(x')^{-1} g(x) \sigma_\alpha(x)$, where $x \in U_\beta$ and $x' \in U_\alpha$. They have to satisfy $h_{\gamma\delta}(x) = \eta_{\delta\beta}(x') h_{\alpha\beta}(x) \eta_{\alpha\gamma}(x)$ $(x \in U_\gamma, x' \in U_\delta)$, where $\eta_{\alpha\gamma}(x)$ are the transition functions $\eta_{\alpha\gamma}(x) = [\sigma_\alpha(x)]^{-1} \sigma_\gamma(x)$.

[33]The sign conventions of Ref. 21 (which are well established in Poincaré gauge theory) differ from those that arise quite naturally in Secs. X and XI of the present paper. Essentially, the ϵ^A, λ^A, and $\Gamma_i{}^A$ of Ref. 21 are the $-\epsilon^A$, $-\lambda^A$, and $-\Gamma_i{}^A$ of the present work. The signs of a^A and M^A also differ.