Gauge theory of a group of diffeomorphisms. II. The conformal and de Sitter groups

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The extension of Hehl's Poincaré gauge theory to more general groups that include space-time diffeomorphisms is worked out for two particular examples, one corresponding to the action of the conformal group on Minkowski space, and the other to the action of the de Sitter group on de Sitter space, and the effect of these groups on physical fields.

I. INTRODUCTION

In a recent work[1] (which we shall refer to as I) a scheme was developed for gauging a group that contains a group of space-time diffeomorphisms as well as (possibly), internal symmetry groups.

Let G be an $(N+M)$-parameter Lie group possessing an N-parameter subgroup H. Introduce, on an M-dimensional base manifold, a connection Γ_i associated with the group G, considered as a Yang–Mills group, and a set of physical fields ψ belonging to a linear representation of H. Under the simultaneous action of an infinitesimal diffeomorphism $x^i \to x^i - \xi^i$ on the base space, and an infinitesimal (local) action of H, we have

$$\delta\Gamma_i = \xi^j \partial_j \Gamma_i + \Gamma_j \partial_i \xi^j + \partial_i \bar{\epsilon} + [\bar{\epsilon}, \Gamma_i], \tag{1.1}$$

$$\delta\psi = \xi^j \partial_j \psi + \bar{\epsilon}\psi, \tag{1.2}$$

where $\bar{\epsilon}$ is an infinitesimal element of the Lie algebra of H, dependent on position on the base manifold. Of course, in (1.2) the representation of H provided by ψ is implied.

When the curvature

$$G_{ij} = \partial_i \Gamma_j - \partial_j \Gamma_i - [\Gamma_i, \Gamma_j] \tag{1.3}$$

vanishes, those transformations (1.1) and (1.2) that leave invariant a particular solution $\Gamma_i(x)$ of (1.3) constitute an $(N+M)$-parameter group of diffeomorphisms on the base space, isomorphic to G. The finite-dimensional linear representation of H corresponding to the action of H on ψ is thereby extended to the action on ψ of a group G of diffeomorphisms.

The purpose of the present work is to illustrate this idea by two particularly interesting special cases.

When $G = \mathrm{SO}(4,2)$, we obtain the action of the conformal group on Minkowski space together with the appropriate transformation laws for physical fields under the action of the conformal group.[2] When $G = \mathrm{SO}(4,1)$ we obtain the action of the de Sitter group on de Sitter space-time together with the appropriate transformation laws for physical fields. Equations (1.1) and (1.2) in this latter case give rise to the basic transformation laws of Poincaré gauge theory, under the Wigner–Inönü contraction of the de Sitter group to the Poincaré group.

II. THE CONFORMAL GAUGE THEORY

The commutation relations for the generators of SO(4,2) can be displayed in the following form:

$$\begin{aligned}
&[\pi_\alpha, \pi_\beta] = 0,\\
&[\pi_\alpha, S_{\beta\gamma}] = \eta_{\alpha\beta}\pi_\gamma - \eta_{\alpha\gamma}\pi_\beta, \quad [\pi_\alpha, \Delta] = \pi_\alpha,\\
&[\pi_\alpha, \kappa_\beta] = 2(\eta_{\alpha\beta}\Delta - S_{\alpha\beta}),\\
&[S_{\alpha\beta}, S_{\gamma\delta}] = \eta_{\beta\gamma}S_{\alpha\delta} - \eta_{\alpha\gamma}S_{\beta\delta} + \eta_{\alpha\delta}S_{\beta\gamma} - \eta_{\beta\delta}S_{\alpha\gamma},\\
&[S_{\alpha\beta}, \Delta] = 0, \quad [S_{\alpha\beta}, \kappa_\gamma] = \kappa_\alpha\eta_{\beta\gamma} - \kappa_\beta\eta_{\alpha\gamma},\\
&[\Delta, \kappa_\alpha] = \kappa_\alpha, \quad [\kappa_\alpha, \kappa_\beta] = 0,
\end{aligned} \tag{2.1}$$

where $\eta_{\alpha\beta}$ is the Minkowskian metric with signature $(+++-)$.

The connection for SO(4,2) can be written

$$\Gamma_i = e_i{}^\alpha \pi_\alpha + \bar{\Gamma}_i, \tag{2.2}$$

where

$$\bar{\Gamma}_i = \tfrac{1}{2}\Gamma_i{}^{\alpha\beta} S_{\alpha\beta} + \phi_i \Delta + \phi_i{}^\alpha \kappa_\alpha. \tag{2.3}$$

The matrix $(e_i{}^\alpha)$ is assumed to be nonsingular, with inverse $(e_\alpha{}^i)$, which can be regarded as the matrix of components of a tetrad. We may employ these matrices to convert Latin (holonomic) to Greek (anholonomic) indices and vice versa. The Minkowskian metric $\eta_{\alpha\beta}$ will be employed for raising and lowering Latin indices.

The infinitesimal element $\bar{\epsilon}$ of the Lie algebra of H can be written

$$\bar{\epsilon} = \tfrac{1}{2}\epsilon^{\alpha\beta} S_{\alpha\beta} + \zeta\Delta + \zeta^\alpha \kappa_\alpha. \tag{2.4}$$

The transformation law (1.1) then has the explicit forms

$$\delta e_i{}^\alpha = \xi^j \partial_j e_i{}^\alpha + e_j{}^\alpha \partial_i \xi^j - e_i{}^\beta(\epsilon_\beta{}^\alpha + \delta_\beta{}^\alpha \zeta), \tag{2.5}$$

$$\begin{aligned}\delta\bar{\Gamma}_i = {}&\xi^j \partial_j \bar{\Gamma}_i + \bar{\Gamma}_j \partial_i \xi^j + \partial_i \bar{\epsilon} + [\bar{\epsilon}, \bar{\Gamma}_i]\\ &- 2e_i{}^\beta(\zeta_\beta \Delta + \zeta^\alpha S_{\alpha\beta}).\end{aligned} \tag{2.6}$$

Observe that the tetrad undergoes Lorentz rotation and dilation under the action of H. Observe also that, due to the final term in (2.6), $\bar{\Gamma}_i$ is *not* a connection for the group H.

At this stage it is possible to impose a metric on the base space (space-time) in a natural way. We define the space-time metric to be the one with respect to which the tetrad is orthonormal:

$$g_{ij} = e_i{}^\alpha e_j{}^\beta \eta_{\alpha\beta}. \tag{2.7}$$

Under the local action of H, this metric responds according to

$$\delta g_{ij} = -2\zeta g_{ij}. \tag{2.8}$$

Thus, the subgroup of SO(4,2) generated by Δ can be identified as Weyl's group of scale transformations.

It is also possible to impose a holonomic linear connection on space-time. We introduce the generalized derivative of the tetrad field (see I):

$$D_i e_j{}^\alpha = \partial_i e_j{}^\alpha + e_j{}^\beta \Gamma_{i\beta}{}^\alpha + e_j{}^\alpha \phi_i, \tag{2.9}$$

and then define

$$\Gamma_{ij}{}^k = e_\alpha{}^k D_i e_j{}^\alpha. \tag{2.10}$$

The $\Gamma_{ij}{}^k$ transform under space-time diffeomorphisms like the components of a linear connection. Moreover, it is a metric-compatible connection:

$$\partial_k g_{ij} - \Gamma_{ki}{}^l g_{lj} - \Gamma_{kj}{}^l g_{il} = 0. \tag{2.11}$$

Under the action of H, it has the transformation law

$$\delta\Gamma_{ij}{}^k = 2(\zeta^k g_{ij} - \delta_i{}^k \zeta_j - \delta_j{}^k \zeta_i). \tag{2.12}$$

We now consider the limiting case with vanishing SO(4,2) curvature:

$$G_{ij} = 0. \tag{2.13}$$

The coordinate system and H-gauge can then be chosen so that

$$e_i{}^\alpha = \delta_i{}^\alpha, \quad \overline{\Gamma}_i = 0. \tag{2.14}$$

We then see from (2.7) that the space-time has become Minkowskian. In this reference system, the distinction between Latin and Greek indices is lost and the conditions for the transformations (1.1) to preserve the relations (2.14) are

$$\begin{aligned} &\partial_\gamma \xi^\alpha = \epsilon_\gamma{}^\alpha + \zeta \delta_\gamma{}^\alpha, \\ &\partial_\gamma \epsilon^{\alpha\beta} = 2(\delta_\gamma{}^\beta \zeta^\alpha - \delta_\gamma{}^\alpha \zeta^\beta), \\ &\partial_\gamma \zeta = 2\zeta_\gamma, \quad \partial_\gamma \zeta^\alpha = 0, \end{aligned} \tag{2.15}$$

[cf. Eqs. (8.2) of I]. The integration of these equations is straightforward. We get, successively,

$$\begin{aligned} &\zeta^\alpha = c^\alpha, \quad \zeta = 2c_\alpha x^\alpha + \rho, \\ &\epsilon^{\alpha\beta} = 2(x^\beta c^\alpha - x^\alpha c^\beta) + \omega^{\alpha\beta}, \end{aligned} \tag{2.16}$$

and finally

$$\xi^\alpha = a^\alpha + x_\gamma \omega^{\gamma\alpha} + \rho x^\alpha + 2x^\alpha c\cdot x - c^\alpha x^2, \tag{2.17}$$

where $a^\alpha, \rho, \omega^{\alpha\beta}$, and c^α are constants of integration; x^2 and $c\cdot x$ denote $x^\alpha x^\beta \eta_{\alpha\beta}$ and $c^\alpha x^\beta \eta_{\alpha\beta}$, respectively. We recognize that the diffeomorphisms $x^\alpha \to x^\alpha - \xi^\alpha$ are the infinitesimal conformal mappings on Minkowski space-time.

The transformation law (1.2) for a field ψ becomes

$$\delta\psi = \xi^\alpha \partial_\alpha \psi + (\tfrac{1}{2}\epsilon^{\alpha\beta} S_{\alpha\beta} + \zeta\Delta + \zeta^\alpha \kappa_\alpha)\psi, \tag{2.18}$$

with $\xi^\alpha, \epsilon^{\alpha\beta}, \zeta$, and ζ^α given by (2.16) and (2.17). Thus we have precisely the transformation law of a physical field on Minkowski space-time, under the action of infinitesimal conformal transformations[2]:

$$\begin{aligned} \delta\psi = [&a^\alpha \partial_\alpha + \tfrac{1}{2}\omega^{\alpha\beta}(S_{\alpha\beta} + x_\alpha \partial_\beta - x_\beta \partial_\alpha) \\ &+ \rho(\Delta + x^\alpha \partial_\alpha) + c^\alpha(\kappa_\alpha + 2(x_\alpha \Delta + x^\beta S_{\alpha\beta}) \\ &+ 2(x_\alpha x^\beta - x^2 \delta_\alpha^\beta)\partial_\beta)]\psi. \end{aligned} \tag{2.19}$$

The reverse of the procedure carried out above is to start with the (global) action of the conformal group on Minkowski space and on fields ψ [given by (2.17) and (2.19)] and then "gauge" the group by making the parameters space-time dependent and introducing auxiliary fields. That is, the conformal group can be gauged in a manner analogous to Kibble's[3,4] gauging of the Poincaré group. The details have been presented elsewhere.[5]

III. THE DE SITTER GAUGE THEORY

The commutation relations for the generators of SO(4,1) can be displayed in the form

$$\begin{aligned} &[\pi_\alpha, \pi_\beta] = -\kappa S_{\alpha\beta}, \\ &[\pi_\alpha, S_{\beta\alpha}] = \eta_{\alpha\beta}\pi_\gamma - \eta_{\alpha\gamma}\pi_\beta, \\ &[S_{\alpha\beta}, S_{\gamma\delta}] = \eta_{\beta\gamma}S_{\alpha\delta} - \eta_{\alpha\gamma}S_{\beta\delta} + \eta_{\alpha\delta}S_{\beta\gamma} - \eta_{\beta\delta}S_{\alpha\gamma}. \end{aligned} \tag{3.1}$$

The subgroup H, generated by the $S_{\alpha\beta}$, is just the Lorentz group. The constant κ is inserted so that the Poincaré group can be regarded as a limiting case.

Introduce the connection

$$\Gamma_i = e_i{}^\alpha \pi_\alpha + \tfrac{1}{2}\Gamma_i{}^{\alpha\beta} S_{\alpha\beta}. \tag{3.2}$$

The transformation law (1.1) becomes

$$\delta e_i{}^\alpha = \xi^j \partial_j e_i{}^\alpha + e_j{}^\alpha \partial_i \xi^j - e_i{}^\beta \epsilon_\beta{}^\alpha, \tag{3.3}$$

$$\begin{aligned} \delta\Gamma_{i\alpha}{}^\beta = &\xi^j \partial_j \Gamma_{i\alpha}{}^\beta + \Gamma_{j\alpha}{}^\beta \partial_i \xi^j + \partial_i \epsilon_\alpha{}^\beta \\ &+ \epsilon_\alpha{}^\gamma \Gamma_{i\gamma}{}^\beta - \Gamma_{i\alpha}{}^\gamma \epsilon_\gamma{}^\beta \end{aligned} \tag{3.4}$$

(in which the Minkowskian metric has been used for raising and lowering Greek indices). Observe that the tetrad is Lorentz rotated by the action of H and that $\Gamma_i{}^{\alpha\beta}$ transforms like a connection for the Lorentz group. We shall employ the symbol D_i to denote the corresponding covariant differentiation. For example,

$$D_i\psi = \partial_i\psi - \tfrac{1}{2}\Gamma_i{}^{\alpha\beta} S_{\alpha\beta}\psi$$

and

$$D_i e_j{}^\alpha = \partial_i e_j{}^\alpha + e_j{}^\beta \Gamma_{i\beta}{}^\alpha.$$

The Lorentz torsion and Lorentz curvature are defined by

$$F_{ij}{}^\alpha = D_i e_j{}^\alpha - D_j e_i{}^\alpha, \tag{3.5}$$

and

$$F_{ij\alpha}{}^\beta = \partial_i \Gamma_{j\alpha}{}^\beta - \partial_j \Gamma_{i\alpha}{}^\beta - \Gamma_{i\alpha}{}^\gamma \Gamma_{j\gamma}{}^\beta + \Gamma_{j\alpha}{}^\gamma \Gamma_{i\gamma}{}^\beta. \tag{3.6}$$

The SO(4,1) curvature is

$$G_{ij} = F_{ij}{}^\alpha \pi_\alpha + \tfrac{1}{2}(F_{ij}{}^{\alpha\beta} + 2\kappa e_i{}^\alpha e_j{}^\beta) S_{\alpha\beta}. \tag{3.7}$$

A holonomic metric and holonomic connection on space-time can be constructed in a natural way from the SO(4,1) connection coefficients. We define

$$g_{ij} = e_i{}^\alpha e_j{}^\beta \eta_{\alpha\beta} \tag{3.8}$$

and

$$\Gamma_{ij}{}^k = e_\alpha{}^k D_i e_j{}^\alpha. \tag{3.9}$$

The connection (3,9) is metric compatible, that is,

$$\partial_i g_{jk} - \Gamma_{ij}{}^l g_{lk} - \Gamma_{ik}{}^l g_{jl} = 0. \tag{3.10}$$

It is not, in general, symmetric:

$$\Gamma_{ij}{}^k - \Gamma_{ij}{}^k = F_{ij}{}^k. \tag{3.11}$$

Thus, the definitions (3.8) and (3.9) impose on the space-time a U(4) structure[6] [in fact, as is apparent from (3.3) and (3.4), the gauged de Sitter group and the gauged Poincaré group are identical].

Now consider the limiting case in which the SO(4,1) curvature (3.7) vanishes. The torsion then vanishes, so the connection (3.9) becomes the Christoffel connection

$$\Gamma_{ij}{}^{k} = \begin{Bmatrix} k \\ ij \end{Bmatrix}. \tag{3.12}$$

The Lorentz curvature (which is now just an anholonomic version of the Riemann tensor constructed from g_{ij}) does not vanish. We have

$$R_{ij}{}^{kl} = F_{ij}{}^{kl} = \kappa(\delta_i{}^l\delta_j{}^k - \delta_j{}^l\delta_i{}^k). \tag{3.13}$$

Thus, the space-time has become a *space of constant curvature*. We can therefore choose the coordinate system to be a system of stereographic coordinates for which

$$g_{ij} = \sigma^2\,\eta_{ij}, \tag{3.14}$$

$$\sigma = 1/[1 + (\kappa x^2/4)], \quad x^2 = \eta_{ij}x^ix^j. \tag{3.15}$$

We can then take the tetrad components to be

$$e_i{}^{\alpha} = \sigma\delta_i{}^{\alpha}. \tag{3.16}$$

It is convenient from now on to convert Latin indices to Greek indices, and vice versa, by means of δ_i^α rather than e_i^α. With this understood, the Lorentz connection determined by (3.9) and (3.12) turns out to be

$$\Gamma_i{}^{\alpha\beta} = \kappa\sigma\delta_i^{[\alpha}x^{\beta]}. \tag{3.17}$$

The transformations (3.3) and (3.4) that leave unchanged these particular functional forms for the tetrad and Lorentz connection are those with parameters ξ^i and $\epsilon^{\alpha\beta}$ satisfying

$$\partial^\alpha\xi^\beta - \tfrac{1}{2}\kappa\sigma\xi\cdot x\eta^{\alpha\beta} - e^{\alpha\beta} = 0, \tag{3.18}$$

and

$$\partial_i\epsilon^{\alpha\beta} + \tfrac{1}{2}\kappa\sigma[\delta_i{}^\alpha(\xi^\beta + \epsilon^{\beta\gamma}x_\gamma) - \delta_i^\beta(\xi^\alpha + \epsilon^{\alpha\gamma}x_\gamma)] = 0. \tag{3.19}$$

Fortunately, we already have partial knowledge about the solution of these equations. The diffeomorphisms that preserve the de Sitter metric are the de Sitter transformations, which, in terms of the stereographic coordinate system, have the infinitesimal form $x^\alpha \to x^\alpha - \xi^\alpha$, where

$$\xi^\alpha = x_\beta\omega^{\beta\alpha} + a^\alpha(1 - (\kappa x^2/4)) + (\kappa/2)x^\alpha a\cdot x; \tag{3.20}$$

$\omega^{\alpha\beta}$ and a^α being the (constant) parameters of the group. Substituting this expression into (3.18) gives

$$\epsilon^{\alpha\beta} = \omega^{\alpha\beta} + (\kappa/2)(a^\alpha x^\beta - a^\beta x^\alpha). \tag{3.21}$$

It is then not difficult to check that Eq. (3.19) is also satisfied.

Equation (1.2) now gives the transformation law for a physical field (belonging to a representation of the Lorentz group) under the action of an infinitesimal de Sitter transformation on a de Sitter space-time:

$$\delta\psi = a^\alpha\left[\left(1 - \frac{\kappa x^2}{4}\right)\partial_\alpha + \frac{\kappa}{2}x_\alpha x^\beta\partial_\beta + \frac{\kappa}{2}S_{\alpha\beta}x^\beta\right]\psi$$
$$+ \tfrac{1}{2}\omega^{\alpha\beta}[x_\alpha\,\partial_\beta - x_\beta\,\partial_\alpha + S_{\alpha\beta}]\psi. \tag{3.22}$$

IV. CONCLUDING REMARKS

Many attempts to construct a gauge theory of a space-time symmetry group encounter difficulties and complications. The reader is referred to the review article of Ivanenko and Sardanashvily[7] and the references cited therein. The difficulties arise from attempting a too close analogy with the pattern established by gauge theories of internal symmetries; if the whole of a space-time group G is "gauged" in the Yang–Mills sense, the gauged "internal translations" destroy the possibility of identifying the translational gauge potentials with a tetrad.[7,8] In our view, in a correct approach to gauging a space-time symmetry G, only the subgroup H is localized in the Yang–Mills sense; the gauged generalization of G in our scheme consists of a local action of H together with *general diffeomorphisms* [or alternatively, general coordinate transformations (GCT)] on space-time M. This viewpoint is already implicit in the de Sitter gauge theory of MacDowell and Mansouri,[9] where invariance of the Lagrangian only under local Lorentz transformations and GCT was imposed. The geometrical background to the MacDowell and Mansouri de Sitter gauge theory corresponds to our scheme [where G is the de Sitter group or its covering group Sp(2,2)].

That the local action of H together with general diffeomorphisms (or GCT) on M does indeed constitute a true gauge theory of a space-time group G is fully justified only when one has shown that the limiting case of "ungauged" transformations does in fact correspond to the correct global action of G on M and on fields in M. The purpose of this work was to demonstrate that this is so for the conformal group, the de Sitter group, and (by Wigner–Inönü contraction of the de Sitter case) the Poincaré group. The "ungauged" limit of Poincaré gauge theory was obtained by Hehl.[6] The gauging of the affine group in accordance with our scheme has been presented elsewhere.[9]

The transformation laws for the points x of M and the matter fields ψ on M, under the global ("ungauged") action of G constitute essentially a *nonlinear realization* of G in the sense of Coleman, Wess, and Zumino[10] or Salam and Strathdee.[11] However, *space-time itself takes the place of the Goldstone fields*, so the usual dynamics of nonlinear realization (Higgs mechanism, spontaneous symmetry breakdown) is not called into play. Thus, our scheme differs radically from that of the Poincaré and de Sitter gauge theories of Tseytlin,[12] in which the whole of G rather than just H acts "internally," but the usual difficulties associated with such a scheme are avoided by realizing the translations nonlinearly. This nonlinear realization of G is associated with spontaneous symmetry breakdown, the broken symmetries being the internal translations.

The relationship between our approach to the gauging of space-time symmetries and that of other approaches becomes clearer when our scheme is expressed in the language of fiber bundles. It is clear from our foregoing remarks that only the subgroup H should act on the fibers, not the whole of G ("no internal translation"). The simplest and most natural translation of our scheme into fiber bundle language consists of expressing the gauge theory of a group G involving space-time and internal symmetries in terms of the group manifold G; specifically, in terms of the principal fiber bundle $G(G/H,H)$ where the coset space G/H is space-time[13] (note that H, not G, is the structural group). This aspect will be dealt with in a subsequent paper.

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