Gauge theory of a group of diffeomorphisms. I. General principles

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Any $(N+M)$-parameter Lie group G with an N-parameter subgroup H can be realized as a global group of diffeomorphisms on an M-dimensional base space B, with representations in terms of transformation laws of fields on B belonging to linear representations of H. The gauged generalization of the global diffeomorphisms consists of general diffeomorphisms (or coordinate transformations) on a base space together with a local action of H on the fields. The particular applications of the scheme to space-time symmetries is discussed in terms of Lagrangians, field equations, currents, and source identities.

I. INTRODUCTION

The theory of Yang and Mills[1] provides a prescription for "gauging" an internal symmetry group. The linear action of the group on physical fields is generalized from a "global" action to a "local" action by the introduction of auxiliary fields—the so-called Yang–Mills potentials or connection coefficients for the group in question. This idea was applied to the group of Lorentz rotations of an orthonormal tetrad (in a metric space-time) by Utiyama[2] and Sciama.[3] It is natural to regard the connection coefficients in this case as the anholonomic components of the linear connection of the space-time. The holonomic linear connection is then metric compatible and asymmetric (the space-time is a U_4, in Hehl's terminology[4]).

The Poincaré group (group of isometries of Minkowski space-time) lies outside the scope of the original Yang–Mills theory because it acts on the space-time as well as on physical fields. Nevertheless, as was shown by Kibble,[5] it can be gauged. The auxiliary fields consist of a tetrad and a connection for the Lorentz rotations of the tetrad. The action of the gauged Poincaré group is the action of general coordinate transformations (or, interpreted actively, space-time diffeomorphisms) together with Lorentz rotations of the tetrad. It is natural then to define the space-time metric to be the one with respect to which the tetrad is orthonormal and to define the linear connection of the space-time to be the one arising from the Lorentz connection. We obtain again the U_4 theory of Utiyama and Sciama. However, the tetrad and the general coordinate transformations arise out of the gauge principle in Kibble's approach; moreover, the metric (and not just the linear connection) is constructed from the auxiliary fields—in the Lorentz gauge theory of Utiyama and Sciama, the tetrad, the metric, and the general coordinate transformations were presupposed *ab initio* and were extraneous to the gauge principle.

The effect of an infinitesimal space-time diffeomorphism (or general coordinate transformation) on an anholonomic field (i.e., a set of scalars that transform linearly and homogeneously under tetrad rotations) is the same as the effect of an infinitesimal *parallel transport* combined with an infinitesimal tetrad rotation. The work of von der Heyde[6] and the subsequent developments of Poincaré gauge theory by Hehl and co-workers[4] have revealed that it is this parallel transport action, rather than space-time diffeomorphisms or coordinate transformations, that should be regarded as the translational part of the gauged Poincaré action. With this interpretation, the tetrad is itself a set of Yang–Mills potentials, constituting the connection for the translational subgroup. The Yang–Mills "field strengths" are the torsion (translational part) and the curvature (rotational part) of the U_4.

Kibble's approach can be applied to more general groups. Some general aspects of the gauging of space-time diffeomorphisms have been worked out by Harnad and Pettitt.[7]

As was shown by Lord,[8] the gauge theory of the affine group, together with a space-time metric imposed as an extraneous field, is equivalent to a purely holonomic metric-affine theory.[9] Of course, the affine extension of the action of the Poincaré group on Minkowski space-time cannot be a symmetry group for a physical theory, but that does not rule out the possibility of the existence of gauge potentials for the group. Indeed, there are some indications that the affine extension of Poincaré gauge theory may be the correct extension required for an understanding of the relationship between strong and gravitational interactions.[10] Poincaré gauge theory has been treated as a limiting case of a de Sitter gauge theory in the work of MacDowell and Mansouri.[11] The gauging of the conformal group (group of diffeomorphisms of Minkowski space-time that preserve the light-cone structure) is usually discussed in the language of fiber bundles, employing second-order frames.[12] As will become clear from the present work, the concept of second-order frames is by no means essential in a gauge theory of the conformal group.

Apart from the extensions mentioned above, there is also the interesting possibility of extending the Poincaré gauge theory so as to include internal symmetries in a nontrivial way.[13]

For a more exhaustive survey of the literature on Poincaré gauge theory and its extensions, the reader is referred to

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the review article of Ivanenko and Sardanashvily.[14]

We shall present a general geometrical framework that includes the above-mentioned theories as particular cases. The central idea is the following: Let G be an $(N+M)$-parameter Lie group possessing an N-parameter subgroup H. Let ψ be a set of fields on an M-dimensional space B, belonging to a linear representation of H. Then G can be realized as a global group of diffeomorphisms on B together with a representation in terms of transformation laws for ψ. Moreover, this can be done in such a way that the gauging of the global diffeomorphism group leads to a local transformation group consisting of general diffeomorphisms on B together with an intrinsic action of H on ψ.

Some aspects of our formalism have been foreshadowed in the work of Harnad and Pettitt.[7] However, the present work goes beyond the scheme of Harnad and Pettitt in several respects, and our approach is different. We do not begin with a global group of diffeomorphisms and attack the problem of gauging it—we begin with a full-fledged gauge theory and come to a global diffeomorphism group as a limiting case.

II. STRUCTURE OF THE GROUP AND ITS POTENTIALS

Consider an $(N+M)$-parameter Lie group G with generators π_α, G_a satisfying the commutation relations

$$\begin{aligned}
&[\pi_\alpha,\pi_\beta] = c_{\alpha\beta}{}^{\gamma}\pi_\gamma + c_{\alpha\beta}{}^{c}G_c,\\
&[\pi_\alpha,G_b] = c_{ab}{}^{\gamma}\pi_\gamma + c_{ab}{}^{c}G_c,\\
&[G_a,G_b] = c_{ab}{}^{c}G_c.
\end{aligned}\tag{2.1}$$

The Greek indices $\alpha,\beta,\ldots$ are M-fold indices and the Latin indices $a,b,\ldots$ are N-fold. The N-parameter subgroup generated by the G_a will be called H. We shall also employ $(N+M)$-fold indices $A,B,\ldots$, in terms of which (2.1) is

$$[G_A,G_B] = c_{AB}{}^{C}G_C \tag{2.2}$$

(where $\pi_\alpha = G_\alpha$, $c_{ab}{}^{\gamma}=0$).

We shall set up a gauge theory of the group G on an M-dimensional base space. We shall take $M=4$ with a view to the physical applications in which the base space is space-time. However, it should be borne in mind that the geometrical framework is valid for any M and thus has potentially wider applications.

To begin with, we regard G as a group that acts on fields over space-time but not on space-time points. The infinitesimal action of G on a field Ψ belonging to a linear representation of G will be written

$$\delta\Psi = \epsilon\Psi, \tag{2.3}$$

where

$$\epsilon = \epsilon^A G_A = \epsilon^\alpha\pi_\alpha + \epsilon^a G_a. \tag{2.4}$$

Here, π_α and G_a denote the matrix representatives of the corresponding generators.

The group G is gauged in the standard Yang–Mills way be introducing Yang–Mills potentials, which are the coefficients of a connection

$$\Gamma_i = \Gamma_i{}^A G_A = e_i{}^\alpha\pi_\alpha + \Gamma_i{}^a G_a. \tag{2.5}$$

The Latin letters $i,j,\ldots$ will be used for holonomic space-time indices. The covariant derivative of a field Ψ,

$$\nabla_i\Psi = \partial_i\Psi - \Gamma_i\Psi, \tag{2.6}$$

transforms like Ψ under the action of an element of G with space-time-dependent parameters provided the connection has the transformation law

$$\delta\Gamma_i = \nabla_i\epsilon = \partial_i\epsilon - [\Gamma_i,\epsilon], \tag{2.7}$$

which corresponds to the transformation law

$$\delta\Gamma_i{}^A = \nabla_i\epsilon^A = \partial_i\epsilon^A + \epsilon^B\Gamma_i{}^C c_{BC}{}^A, \tag{2.8}$$

for the Yang–Mills potentials. The Yang–Mills field strengths are the coefficients of the curvature

$$\begin{aligned}
G_{ij} &= \partial_i\Gamma_j - \partial_j\Gamma_i - [\Gamma_i,\Gamma_j]\\
&= G_{ij}{}^A G_A = G_{ij}{}^\alpha\pi_\alpha + G_{ij}{}^a G_a,
\end{aligned}\tag{2.9}$$

which leads to

$$G_{ij}{}^A = \partial_i\Gamma_j{}^A - \partial_j\Gamma_i{}^A - \Gamma_i{}^B\Gamma_j{}^C C_{BC}{}^A. \tag{2.10}$$

The fields strengths have a linear homogeneous transformation law

$$\delta G_{ij} = [\epsilon,G_{ij}], \quad \delta G_{ij}{}^A = \epsilon^B G_{ij}{}^C c_{BC}{}^A \tag{2.11}$$

and satisfy the Bianchi identities

$$\nabla_{[i}G_{jk\,]}{}^A = 0. \tag{2.12}$$

Now let ψ be a field belonging to a linear representation R of the subgroup H. We write the infinitesimal transformation law of ψ under the action of H in the form

$$\delta\psi = \epsilon^a\overline{G}_a\psi, \tag{2.13}$$

where the $\overline{G}_a$ are the matrix representatives of the generators, in the representation R. *In general, it is not possible to extend R to a representation of G.* Three particular representations of H, deducible from the structure constants of G, are an $(N+4)$-dimensional representation T, a four-dimensional representation S, and an N-dimensional representation C (the adjoint representation of H), generated, respectively, by the matrices T_a, S_a, and C_a defined by

$$(T_a)_B{}^C = C_{Ba}{}^C, \quad (S_a)_\beta{}^\gamma = c_{\beta a}{}^\gamma, \quad (C_a)_b{}^c = c_{ba}{}^c. \tag{2.14}$$

(The relations $[T_a,T_b] = c_{ab}{}^cT_c$, $[S_a,S_b] = c_{ab}{}^cS_c$, and $[C_a,C_b] = c_{ab}{}^cC_c$ are consequences of the Jacobi identities for the generators of G.)

Observe that if

$$c_{ab}{}^c = 0, \tag{2.15}$$

the representation T is just the direct sum of the representations S and C. The relation (2.15) holds, for instance, when G is the Poincaré group, the affine group, or the de Sitter group. If (2.15) does not hold, then T is reducible but not completely reducible. An example of this is $G = \mathrm{SO}(4,2)$, a circumstance that leads to interesting special features for the gauge theory of the conformal group.

The infinitesimal transformation laws for fields belonging to the representations T, S, and C of H, under the action of H, are, respectively,

$$\begin{aligned}
&\delta\phi_A = \epsilon^b c_{Ab}{}^C\phi_C, \quad \delta\chi_\alpha = \epsilon^b c_{ab}{}^\gamma\chi_\gamma,\\
&\delta\psi_a = \epsilon^b c_{ab}{}^c\psi_c.
\end{aligned}\tag{2.16}$$

The contragredient representations $\widetilde{T}$, $\widetilde{S}$, and $\widetilde{C}$ have the corresponding infinitesimal transformation laws

$$\delta\phi^A = -\epsilon^b\phi^C c_{Cb}{}^A,\ \delta\chi^\alpha = -\epsilon^b\chi^\gamma c_{\gamma b}{}^\alpha, \qquad (2.17)$$
$$\delta\psi^a = -\epsilon^b\psi^c c_{cb}{}^a.$$

Observe in particular that, if χ_α transforms according to S, we can extend it to a χ_A transforming according to T, simply by defining $\chi_a = 0$.

Under the action of H, the Yang–Mills potentials of G have the transformation laws

$$\delta e_i{}^\alpha = -\epsilon^b e_i{}^\gamma c_{\gamma b}{}^\alpha, \qquad (2.18)$$

$$\delta\Gamma_i{}^a = \partial_i\epsilon^a + \epsilon^b\Gamma_i{}^c c_{bc}{}^a - \epsilon^b e_i{}^\gamma c_{\gamma b}{}^a. \qquad (2.19)$$

In regions of space-time where the 4×4 matrix $(e_i{}^\alpha)$ is nonsingular, its inverse $(e_\alpha{}^i)$ specifies a *tetrad field*, which belongs to the representation S of H:

$$\delta e_\alpha{}^i = \epsilon^b c_{\alpha b}{}^\gamma e_\gamma{}^i. \qquad (2.20)$$

We shall employ these matrices to convert holonomic indices $i,j,...$ to anholonomic indices $\alpha,\beta,...$ (which are associated with the representations S and $\widetilde{S}$ of H), in the usual way. For example, $\chi_i = e_i{}^\alpha\chi_\alpha$ and $\chi^i = \chi^\alpha e_\alpha{}^i$ are space-time vectors, invariant under H.

The third term in (2.19) shows that in general the $\Gamma_i{}^a$ are *not* Yang–Mills potentials for H; they are Yang–Mills potentials only if the condition (2.15) holds. Nevertheless, we shall employ the $\Gamma_i{}^a$ to define a *parallel transport* of a field, and an associated *generalized derivative*

$$D_i\psi = \partial_i\psi - \overline{\Gamma}_i\psi,\quad \overline{\Gamma}_i = \Gamma_i{}^a\overline{G}_a, \qquad (2.21)$$

which is not a true covariant derivative (transforming like ψ under the action of H) unless (2.15) holds. Nevertheless, it plays a crucial role in our gauge theory of the group G. We have shown elsewhere[15] how such a noncovariant derivative arises when the conformal group is gauged following Kibble's method of gauging the Poincaré group. In Sec. IV, we shall see that D_i is actually a constituent of a "generalized covariant derivative."

The transformation law of $D_i\psi$ under the action of H is easily found. It is

$$\delta D_i\psi = \epsilon^b(\overline{G}_b D_i\psi + e_i{}^\gamma c_{\gamma b}{}^a\overline{G}_a\psi). \qquad (2.22)$$

The rule for constructing the generalized derivative D_iX of a field variable is to subtract from ∂_iX the expression obtained by replacing ϵ^b by $-\Gamma_i{}^b$ in the infinitesimal change δX brought about by the action of H. Applying this rule, we find that

$$[D_i,D_j]\psi = -F_{ij}{}^a\overline{G}_a\psi, \qquad (2.23)$$

where

$$F_{ij}{}^a = \partial_i\Gamma_j{}^a - \partial_j\Gamma_i{}^a - \Gamma_i{}^b\Gamma_j{}^c c_{bc}{}^a$$
$$+ (e_j{}^\gamma\Gamma_i{}^b - e_i{}^\gamma\Gamma_j{}^b)c_{\gamma b}{}^a. \qquad (2.24)$$

This important quantity will be called *H-curvature*. Equally important is the *H-torsion* defined by

$$D_ie_j{}^\alpha - D_je_i{}^\alpha = F_{ij}{}^\alpha. \qquad (2.25)$$

That is,

$$F_{ij}{}^\alpha = \partial_ie_j{}^\alpha - \partial_je_i{}^\alpha + (e_j{}^\gamma\Gamma_i{}^b - e_i{}^\gamma\Gamma_j{}^b)c_{\gamma b}{}^\alpha. \qquad (2.26)$$

Comparison of (2.10) with (2.24) and (2.26) shows that the H-curvature and H-torsion are related to the field strengths for G through the relations

$$G_{ij}{}^A = F_{ij}{}^A - e_i{}^\beta e_j{}^\gamma c_{\beta\gamma}{}^A. \qquad (2.27)$$

In particular, if the π_α generate an Abelian subgroup of G (as is the case, for example, for the Poincaré group and conformal group), then $G_{ij}{}^A = F_{ij}{}^A$.

III. SPACE-TIME DIFFEOMORPHISMS

The Yang–Mills potentials $\Gamma_i{}^A$ transform as a covariant vector under general coordinate transformations or space-time diffeomorphisms. That is, under an infinitesimal space-time diffeomorphism $x^i \to x^i - \xi^i$, combined with an infinitesimal action of the group G,

$$\delta\Gamma_i{}^A = \xi^i\partial_j\Gamma_i{}^A + \Gamma_j{}^A\partial_i\xi^j + \nabla_i\epsilon^A \qquad (3.1)$$

[where δ denotes the substantial variation, $\delta X = X'(x) - X(x)$]. In terms of the new parameters

$$\lambda^A = \epsilon^A + \xi^i\Gamma_i{}^A, \qquad (3.2)$$

this is just

$$\delta\Gamma_i{}^A = \xi^jG_{ij}{}^A + \nabla_i\lambda^A. \qquad (3.3)$$

We now link the action of the infinitesimal generators π_α to the space-time diffeomorphisms by making the identification

$$\lambda^\alpha = \xi^\alpha. \qquad (3.4)$$

This step is of central importance in our approach. The gauge group G now has an action on the space-time points as well as on field components. Equation (3.3) now becomes

$$\delta\Gamma_i{}^A = \lambda^\beta F_{\beta i}{}^A + D_i\lambda^A + \lambda^b e_i{}^\gamma c_{b\gamma}{}^A, \qquad (3.5)$$

or, more explicitly,

$$\delta e_i{}^\alpha = \lambda^\beta F_{\beta i}{}^\alpha + \partial_i\lambda^\alpha + (\lambda^b e_i{}^\gamma - \lambda^\gamma\Gamma_i{}^b)c_{b\gamma}{}^\alpha, \qquad (3.6)$$

$$\delta\Gamma_i{}^a = \lambda^\beta F_{\beta i}{}^a + \partial_i\lambda^a + \lambda^b\Gamma_i{}^c c_{bc}{}^a$$
$$+ (\lambda^b e_i{}^\gamma - \lambda^\gamma\Gamma_i{}^b)c_{b\gamma}{}^a. \qquad (3.7)$$

This change is exactly the change brought about by a space-time diffeomorphism combined with an action of H [as is obvious from the fact that (3.4) is $\epsilon^\alpha = 0$]. It can therefore be associated with the change

$$\delta\psi = \xi^i\partial_i\psi + \epsilon^a\overline{G}_a\psi \qquad (3.8)$$

in a field ψ belonging to a linear representation of H, scalar under the diffeomorphisms. In terms of the new parameters (3.2), this is

$$\delta\psi = \lambda^\alpha D_\alpha\psi + \lambda^a\overline{G}_a\psi. \qquad (3.9)$$

In this form, we see that the action of the generators π_α is associated with *parallel transport* of ψ.

The transformation laws (3.5) *and* (3.9) *are the fundamental equations in our gauge theory of the group G.*

IV. THE MODIFIED LIE ALGEBRA

From (2.20) and (2.22) we can deduce the transformation law, under H, of the anholonomic generalized derivative $D_\alpha\psi = e_\alpha{}^iD_i\psi$; we find

$$\delta D_\alpha\psi = \epsilon^b(\overline{G}_bD_\alpha\psi + c_{\alpha b}{}^\gamma D_\gamma\psi + c_{\alpha b}{}^c\overline{G}_c\psi). \qquad (4.1)$$

Observe also that, under H,

$$\delta\overline{G}_a\psi = \overline{G}_a\delta\psi = \epsilon^b\overline{G}_a\overline{G}_b\psi = \epsilon^b(\overline{G}_b\overline{G}_a + c_{ab}{}^c G_c)\psi. \quad (4.2)$$

These two transformation laws can be combined in the single expression

$$\delta Q_A\psi = \epsilon^b(\overline{G}_b Q_A\psi + c_{Ab}{}^C Q_C\psi), \quad (4.3)$$

where the generators Q_A are defined by

$$Q_\alpha\psi = -D_\alpha\psi, \quad Q_a\psi = -\overline{G}_a\psi. \quad (4.4)$$

Thus, the components of $Q_A\psi$ transform according to the representation $T\otimes R$ of H.

Now, the Q_A are the operators that generate the changes (3.9) in ψ,

$$\delta\psi = -\lambda^A Q_A\psi. \quad (4.5)$$

We shall now look for the commutation relations satisfied by these operators.

The transformation law of $Q_A\psi$ under H is

$$\epsilon^b Q_b Q_A\psi = -\delta Q_A\psi = -\epsilon^b(\overline{G}_b Q_A\psi + c_{Ab}{}^c Q_c\psi).$$

Hence,

$$Q_b Q_a\psi = (\overline{G}_b\overline{G}_a + c_{ab}{}^c\overline{G}_c)\psi = \overline{G}_a\overline{G}_b\psi.$$

Therefore,

$$[Q_b,Q_a]\psi = [\overline{G}_a,\overline{G}_b]\psi = c_{ab}{}^c\overline{G}_c\psi = c_{ba}{}^c Q_c\psi,$$

establishing that

$$[Q_a,Q_b] = c_{ab}{}^c Q_c. \quad (4.6)$$

Also, from (4.3), we have

$$Q_b Q_\alpha\psi = G_b D_\alpha\psi - c_{\alpha b}{}^C Q_C\psi. \quad (4.7)$$

Since the $\overline{G}_b$ are constant matrices, their generalized derivative vanishes

$$[D_i\overline{G}_b = \partial_i\overline{G}_b - \Gamma_i{}^a(\overline{G}_a\overline{G}_b - \overline{G}_b\overline{G}_a - c_{ab}{}^c\overline{G}_c) = 0].$$

Therefore

$$Q_\alpha Q_b\psi = D_\alpha\overline{G}_b\psi = \overline{G}_b D_\alpha\psi. \quad (4.8)$$

Subtracting (5.8) from (5.7), we find that

$$[Q_\alpha,Q_b] = C_{\alpha b}{}^\gamma Q_\gamma + c_{\alpha b}{}^c Q_c. \quad (4.9)$$

The relations (4.6) and (4.9) for the Q_A are just like the commutation relations (2.1) with which we set out. However, the first commutator (2.1) is modified in a manner already familiar from Poincaré gauge theory.[4] We have

$$\begin{aligned}[D_\alpha,D_\beta]\psi &= e_\alpha{}^i D_i(\epsilon_\beta{}^j D_j\psi) - (\alpha\leftrightarrow\beta)\\ &= (D_\alpha e_\beta{}^j - D_\beta e_\alpha{}^j)D_j\psi + e_\alpha{}^i e_\beta{}^j[D_i,D_j]\psi\\ &= (D_\alpha e_\beta{}^j - D_\beta e_\alpha{}^j)D_j\psi - F_{\alpha\beta}{}^c\overline{G}_c\psi.\end{aligned}$$

But

$$\begin{aligned}F_{\alpha\beta}{}^j &= e_\gamma{}^j e_\alpha{}^i e_\beta{}^k(D_i e_k{}^\gamma - D_k e_i{}^\gamma)\\ &= e_\gamma{}^j(e_\beta{}^k D_\alpha e_k{}^\gamma - e_\alpha{}^i D_\beta e_i{}^\gamma) = D_\beta e_\alpha{}^j - D_\alpha e_\beta{}^j.\end{aligned}$$

Therefore,

$$[D_\alpha,D_\beta] = -F_{\alpha\beta}{}^\gamma D_\gamma - F_{\alpha\beta}{}^c G_c, \quad (4.10)$$

which establishes that

$$[Q_\alpha,Q_\beta] = F_{\alpha\beta}{}^\gamma Q_\gamma + F_{\alpha\beta}{}^c Q_c. \quad (4.11)$$

The H-curvature and H-torsion have taken the place of structure constants, in a modification of the Lie algebra of G.

We shall now write the commutation relations for the Q_A, that we have just found, in the more concise form

$$[Q_A,Q_B] = F_{AB}{}^C Q_C, \quad (4.12)$$

where the $F_{\alpha\beta}{}^C$ are the H-curvature and H-torsion and the remaining components of $F_{AB}{}^C$ are the original structure constants of G. Observe that, if the curvature G_{ij} vanishes, the commutation relations (4.12) reduce to those of the Lie algebra of G [see (2.27)].

The fact that the appropriate derivative operator D_α for the gauge theory of G is not in general a covariant derivative operator is at first sight disturbing. However, recall that the covariant derivative operator associated with a gauge group H is by definition an operator that acts on a field ψ, belonging to a linear homogeneous representation R of H, to produce a derivative of H that transforms linearly and homogeneously. In the present context, such a covariant derivative operator does in fact exist, namely the operator

$$-Q_A = \begin{pmatrix} D_\alpha \\ \overline{G}_a \end{pmatrix}, \quad (4.13)$$

which produces a derivative transforming according to the linear homogeneous representation $T\otimes R$. Thus, we have an interesting extension of the usual notion of covariant differentiation.

Now let ϕ_A and ϕ^A be quantities belonging to the representations T and $\widetilde{T}$ of H. We define a new derivative operator for such quantities, suggested by the relations (4.12) and the usual structure of "covariant derivatives of the adjoint and coadjoint representations of a Lie group":

$$\begin{aligned}\mathscr{D}_i\phi_A &= \partial_i\phi_A - \Gamma_i{}^B F_{AB}{}^C\phi_C,\\ \mathscr{D}_i\phi^A &= \partial_i\phi^A + \phi^C\Gamma_i{}^B F_{CB}{}^A.\end{aligned} \quad (4.14)$$

Unlike the quantities $D_i\phi_A$ and $D_i\phi^A$, we find that these derivatives are true covariant derivatives in that they transform like ϕ_A (resp. ϕ^A) under the action of H. The geometrical significance of the operator $\mathscr{D}_i$ is at present obscure. However, as we shall see, it leads to striking formal simplifications of some of the fundamental relationships of our theory.

V. THE SOURCE IDENTITIES

Any gauge theory has two distinct aspects: the purely geometrical aspect and the physical aspect. The physics is introduced by means of Lagrangians, and, in the case of gauge theories of space-time groups, by means of hypotheses concerning the relationship between the potentials and the metric and affine properties of space-time. In the preceding sections, we have set up the formalism for the geometrical aspect of a gauge theory of the group G. We now relate this to physics by postulating the existence of Lagrangian theories invariant under the action of the gauge transformations.

Suppose there exists a Lagrangian density $\mathscr{L}(\psi, \partial_i\psi,\Gamma_i{}^A)$ whose field equations are form-invariant under the action of the transformations (3.5) and (3.9). The interesting question of what are the possible forms of Lagrangian densities (if any) for a given group G will not be dealt with here; we simply assume the existence of $\mathscr{L}$ and examine the consequences of that assumption.

The covariance requirement is

$$\partial_i(\xi^i\mathscr{L})=\delta\mathscr{L}=\frac{\partial\mathscr{L}}{\partial\psi}\delta\psi+\frac{\partial\mathscr{L}}{\partial\,\partial_i\psi}\delta\partial_i\psi+\frac{\partial\mathscr{L}}{\partial\Gamma_i{}^A}\delta\Gamma_i{}^A. \tag{5.1}$$

Define the *sources* of the Yang–Mills potentials

$$\Sigma_A{}^i=\frac{\partial\mathscr{L}}{\partial\Gamma_i{}^A} \tag{5.2}$$

and employ the field equations

$$0=\frac{\delta\mathscr{L}}{\delta\psi}=\frac{\partial\mathscr{L}}{\partial\psi}-\partial_i\Pi^i,\quad \Pi^i=\frac{\partial\mathscr{L}}{\partial\,\partial_i\psi}. \tag{5.3}$$

We find that

$$\partial_i(\xi^i\mathscr{L}-\Pi^i\delta\psi)=\Sigma_A{}^i\delta\Gamma_i{}^A. \tag{5.4}$$

Now,

$$\xi^i\mathscr{L}-\Pi^i\delta\psi=\lambda^A\theta_A{}^i, \tag{5.5}$$

where

$$\theta_\alpha{}^i=\mathscr{L}e_\alpha{}^i-\Pi^iD_\alpha\psi, \tag{5.6}$$

$$\theta_a{}^i=-\Pi^i\overline{G}_a\psi. \tag{5.7}$$

These quantities are recognizable as a canonical energy-momentum density and a set of (intrinsic) currents associated with the subgroup H. We find that $\theta_A{}^i$ belongs to the representation T of H,

$$\delta\theta_A{}^i=\lambda^b c_{Ab}{}^C\theta_C{}^i \tag{5.8}$$

(and is a vector density under space-time diffeomorphisms).

We now have

$$\partial_i(\lambda^A\theta_A{}^i)=D_i(\lambda^A\theta_A{}^i)$$
$$=\Sigma_A{}^i(\lambda^\alpha F_{\alpha i}{}^A+D_i\lambda^A+\lambda^b e_i{}^\gamma c_{b\gamma}{}^A). \tag{5.9}$$

Equating coefficients of $D_i\lambda^A$ identifies the sources of the Yang–Mills potentials as the canonical currents,

$$\Sigma_A{}^i=\theta_A{}^i. \tag{5.10}$$

Equating coefficients of λ^α and λ^a gives the source identities

$$D_i\theta_\alpha{}^i=\theta_A{}^iF_{\alpha i}{}^A, \tag{5.11}$$

$$D_i\theta_a{}^i=\theta_A{}^ic_{ai}{}^A. \tag{5.12}$$

The first of these is recognizable as a generalized energy-momentum conservation law, the right-hand side being a set of "Lorentz forces" constructed from the currents and field strengths.

The relations (5.10) are explicitly

$$\frac{\partial\mathscr{L}}{\partial e_i{}^\alpha}=\mathscr{L}e_\alpha{}^i-\frac{\partial\mathscr{L}}{\partial\,\partial_i\psi}D_\alpha\psi \tag{5.13}$$

and

$$\frac{\partial\mathscr{L}}{\partial\Gamma_i{}^a}=-\frac{\partial\mathscr{L}}{\partial\,\partial_i\psi}\overline{G}_a\psi. \tag{5.14}$$

They imply that the Lagrangian density has the form

$$\mathscr{L}=eL(\psi,D_\alpha\psi),\quad e=|e_i{}^\alpha|. \tag{5.15}$$

VI. THE FIELD EQUATIONS

Let us now suppose the existence of a Lagrangian density $\mathscr{V}(\Gamma_i{}^A,\partial_j\Gamma_i{}^A)$ for the Yang–Mills potentials, and add it to $\mathscr{L}$. Covariance requirements impose the restriction

$$\delta\mathscr{V}=\partial_i(\xi^i\mathscr{V})=\frac{\partial\mathscr{V}}{\partial\Gamma_i{}^A}\delta\Gamma_i{}^A+\mathscr{H}_A{}^{ij}\delta\,\partial_j\Gamma_i{}^A, \tag{6.1}$$

where

$$\mathscr{H}_A{}^{ij}=\frac{\partial\mathscr{V}}{\partial_j\Gamma_i{}^A}. \tag{6.2}$$

(The possibility of constructing such Lagrangians will be considered in Sec. VII.)

The field equations obtained from variation of $\Gamma_i{}^A$ in $\mathscr{L}+\mathscr{V}$ are

$$\frac{\delta\mathscr{V}}{\delta\Gamma_i{}^A}=\frac{\partial\mathscr{V}}{\partial\Gamma_i{}^A}-\partial_j\mathscr{H}_A{}^{ij}=-\theta_A{}^i. \tag{6.3}$$

Hence

$$\partial_j(\xi^j\mathscr{V}-\mathscr{H}_A{}^{ij}\delta\Gamma_i{}^A)=-\theta_A{}^i\delta\Gamma_i{}^A, \tag{6.4}$$

i.e.,

$$D_j[\lambda^A\mathscr{E}_A{}^j-\mathscr{H}_A{}^{ij}D_i\lambda^A]$$
$$=-\theta_A{}^i(\lambda^\beta F_{\beta i}{}^A+D_i\lambda^A+\lambda^b e_i{}^\gamma c_{b\gamma}{}^A), \tag{6.5}$$

where

$$\mathscr{E}_\alpha{}^j=\mathscr{V}e_\alpha{}^j-\mathscr{H}_A{}^{\beta j}F_{\alpha\beta}{}^A, \tag{6.6}$$

$$\mathscr{E}_a{}^j=-\mathscr{H}_A{}^{\beta j}c_{a\beta}{}^A. \tag{6.7}$$

Equating coefficients of $\partial_i\,\partial_j\lambda^A$ gives

$$\mathscr{H}_A{}^{ij}=-\mathscr{H}_A{}^{ji} \tag{6.8}$$

(which shows that derivatives of the $\Gamma_i{}^A$ have to be contained in $\mathscr{V}$ in the combination $G_{ij}{}^A$), and equating coefficients of $D_j\lambda^A$ gives the field equations for the Yang–Mills potentials in the Maxwellian form

$$D_j\mathscr{H}_A{}^{ij}=\theta_A{}^i+\mathscr{E}_A{}^i. \tag{6.9}$$

Observe that the $\mathscr{E}_A{}^i$ are the energy-momentum density and H-currents for the Yang–Mills potentials. Equation (5.4) can be regarded as a definition of the energy-momentum and the H-currents. Applying this to the Yang–Mills Lagrangian, we find

$$\xi^j\mathscr{V}-\frac{\partial\mathscr{V}}{\partial\,\partial_j\Gamma_i{}^A}\delta\Gamma_i{}^A=\lambda^A\mathscr{E}_A{}^j+\text{terms in }D_i\lambda^A. \tag{6.10}$$

The peculiar derivative operator $\mathscr{D}_i$ introduced in Sec. IV can be used to cast some of our equations into a particularly elegant form. For example, the transformation law (3.5) for the Yang–Mills potentials is just

$$\delta\Gamma_i{}^A=\mathscr{D}_i\lambda^A, \tag{6.11}$$

the source identities (5.11) and (5.12) are

$$\mathscr{D}_i\theta_A{}^i=0 \tag{6.12}$$

and the field equations (6.9) are

$$\mathscr{D}_i\mathscr{H}_A{}^{ij}=\theta_A{}^i+e_A{}^i\mathscr{V} \tag{6.13}$$

(where $e_a{}^i$ is defined to be zero). Note that the expressions on the right and left in (6.12) and (6.13) transform linearly and *homogeneously* under the action of H (they belong to the representation T), but this was not the case for the source identities and field equations as originally given, unless $c_{ab}{}^{c}=0$. Note finally that the definitions (5.6), (5.7), (6.6), and (6.7) of energy-momentum densities and H-currents can be written in the "manifestly covariant" forms

$$\theta_{A}{}^{i}=\mathscr{L}e_{A}{}^{i}+\Pi^{i}Q_{A}\psi, \tag{6.14}$$

$$\mathscr{E}_{A}{}^{i}=\mathscr{V}e_{A}{}^{i}-\mathscr{H}_{B}{}^{\beta i}F_{A\beta}{}^{B}. \tag{6.15}$$

VII. STRUCTURE OF LAGRANGIANS

So far, we have not proposed any particular form for the Lagrange density $\mathscr{V}$. We have found that it has to be a function of $\Gamma_{i}{}^{A}$ and $G_{ij}{}^{A}$. An obvious choice is the Maxwell-type Lagrange density quadratic in curvature,

$$\mathscr{V}=(1/\kappa)g^{1/2}g^{ij}g^{kl}G_{ik}{}^{A}G_{jl}{}^{B}\gamma_{AB}, \tag{7.1}$$

where κ is a constant, the matrix (g_{ij}) is constructed from the tetrad according to

$$g_{ij}=e_{i}{}^{\alpha}e_{j}{}^{\beta}\eta_{\alpha\beta}, \tag{7.2}$$

where $(\eta_{\alpha\beta})$ is some nonsingular symmetric matrix, (g^{ij}) is the inverse of (g_{ij}), and g is its determinant. The matrix (γ_{AB}) is the Cartan form for G,

$$\gamma_{AB}=-c_{EA}{}^{F}c_{FB}{}^{E}. \tag{7.3}$$

Clearly, g_{ij} transforms as a tensor under coordinate transformations or diffeomorphisms of space-time (and can be interpreted as the space-time metric). The function (7.1) is a scalar density under coordinate transformations or space-time diffeomorphisms. It will be invariant under the action of H provided

$$c_{\alpha b}{}^{\gamma}\eta_{\gamma\beta}+c_{\beta b}{}^{\gamma}\eta_{\gamma\alpha}=\tfrac{1}{2}\eta_{\alpha\beta}c_{\gamma\beta}{}^{\gamma}. \tag{7.4}$$

This can be regarded as a restriction on the choice of the group G when η is given. In terms of the four-dimensional representation S of H, it can be written more succinctly as

$$S\eta S^{T}=|S|^{1/2}\eta. \tag{7.5}$$

In the following section we shall show that the gauge theory of the group G that we have set up can be obtained by gauging a global group of space-time diffeomorphisms. This will of course only lead to plausible physics if the global group of diffeomorphisms is related to the geometrical properties of the space-time on which it acts (e.g., Poincaré or conformal transformations on Minkowski space, de Sitter transformations on de Sitter space). The Poincaré gauge theory, de Sitter gauge theory, and conformal gauge theory all have groups G that satisfy a condition of the form (7.5). The affine group does not, but in the affine gauge theory[8] an independent dynamical metric field g_{ij}, unrelated to the tetrad, is introduced into the structure of Lagrangians. Of course, in some cases (7.1) will not be the uniquely possible choice for $\mathscr{V}$. For example, as is well known, in Poincaré gauge theory a curvature scalar is a possible choice,[5] and so are terms quadratic in torsion.[4]

The possible forms for the matter Lagrangian present a more complicated problem, which will not be dealt with here, except to mention that matter Lagrangians with the required transformation laws can be constructed for the gauge theories of the Poincaré, de Sitter, and conformal groups.

The gauge theories of the de Sitter and conformal groups that arise as particular cases of our formalism will be presented in a sequel to the present work.

VIII. REALIZATION OF *G* AS A GLOBAL GROUP OF DIFFEOMORPHISMS

We consider now what happens when the curvature G_{ij} vanishes. We have noted already that the operators Q_A then satisfy the commutation relations of the Lie algebra of G. Let

$$\mathring{\Gamma}_{i}=\Gamma_{i}{}^{A}G_{A} \tag{8.1}$$

be a connection, with vanishing curvature, whose coefficients are given functions of space-time (note that $\mathring{e}_{i}{}^{\alpha}$ is required to be a nonsingular matrix, so $\mathring{\Gamma}_{i}=0$ is not an appropriate choice). The transformations (3.5) that preserve the given form of the $\mathring{\Gamma}_{i}{}^{A}$ are

$$\nabla_{i}\lambda^{A}=0=\partial_{i}\lambda^{A}+\lambda^{C}\mathring{\Gamma}_{i}{}^{D}c_{CD}{}^{A}. \tag{8.2}$$

We have a set of $N+4$ linear differential equations to be solved for the $N+4$ parameters λ^{A}. The integrability conditions are $\mathring{G}_{ij}{}^{A}=0$, which are satisfied. The general solution can be written in terms of an element σ of the group G, satisfying

$$\partial_{i}\sigma+\sigma\mathring{\Gamma}_{i}=0. \tag{8.3}$$

The integrability conditions for these equations are $G_{ij}=0$. Now denote the matrix that represents σ in the adjoint representation by $E_{B}{}^{A}$. Then

$$\partial_{i}E_{B}{}^{A}+E_{B}{}^{C}\mathring{\Gamma}_{i}{}^{D}c_{CD}{}^{A}=0. \tag{8.4}$$

Since the matrix $E_{B}{}^{A}$ is nonsingular, its columns provide $N+4$ linearly independent solutions of (8.2). The general solution is therefore

$$\lambda^{A}=a^{B}E_{B}{}^{A}, \tag{8.5}$$

where the a^{B} are constants.

The transition law of ψ, under these specialized transformations, is

$$\delta\psi=a^{B}M_{B}, \tag{8.6}$$

$$M_{B}=E_{B}{}^{\alpha}D_{\alpha}+E_{B}{}^{a}\overline{G}_{a}. \tag{8.7}$$

In order to establish that (8.6) corresponds to a representation of G, with a^{B} as parameters, we have to show that

$$[M_{A},M_{B}]=c_{AB}{}^{C}M_{C}. \tag{8.8}$$

Let $\lambda^{A}=a^{B}E_{B}{}^{A}$ and $\mu^{A}=b^{B}E_{B}{}^{A}$ be two arbitrary solutions of (8.2). Then, since a^{B} and b^{B} are constants, we have

$$\begin{aligned}a^{A}b^{B}[M_{A},M_{B}]&=[\lambda^{\alpha}D_{\alpha}+\lambda^{a}\overline{G}_{a},\mu^{\beta}D_{\beta}+\mu^{b}\overline{G}_{b}]\\&=(\lambda^{\alpha}D_{\alpha}\mu^{\beta}-(\lambda\leftrightarrow\mu))D_{\beta}+\lambda^{\alpha}\mu^{\beta}[D_{\alpha},D_{\beta}]\\&\quad+(\lambda^{\alpha}D_{\alpha}\mu^{b}-(\lambda\leftrightarrow\mu))\overline{G}_{b}+\lambda^{a}\lambda^{b}[\overline{G}_{a},\overline{G}_{b}].\end{aligned}$$

Now, (8.2) can be written

$$D_{\alpha}\lambda^{A}=-\lambda^{B}c_{B\alpha}{}^{A}, \tag{8.9}$$

and we have an identical equation in μ^{A}. Therefore

$$a^{A}b^{B}[M_{A},M_{B}]=\lambda^{A}\mu^{B}(c_{AB}{}^{\gamma}D_{\gamma}+c_{AB}{}^{c}\overline{G}_{c})$$

(in arriving at this, we make use of $c_{ab}{}^{\gamma}=0$). Therefore

$$[M_A,M_B]=E_A{}^E E_B{}^F(c_{EF}{}^{\gamma}D_\gamma+c_{EF}{}^c\overline{G}_c).$$

Since $E_A{}^B$ belongs to the adjoint representation of G, it satisfies the identity

$$E_A{}^E E_B{}^F c_{EF}{}^C=c_{AB}{}^D E_D{}^C. \tag{8.10}$$

The result (8.8) then follows immediately.

An alternative form for the M_B is

$$M_B=B_B{}^i\partial_i+B_B{}^a\overline{G}_a, \tag{8.11}$$

where the coefficients are defined by

$$E_B{}^\alpha=B_B{}^i\mathring{e}_i{}^\alpha,\quad E_B{}^a=B_B{}^a+B_B{}^i\mathring{\Gamma}_i{}^a. \tag{8.12}$$

The commutation relations (8.8) imply the following identities:

$$B_A{}^i\partial_i B_B{}^j-B_B{}^i\partial_i B_A{}^j=c_{AB}{}^C B_C{}^j, \tag{8.13}$$

$$B_A{}^i\partial_i B_B{}^a-B_B{}^i\partial_i B_A{}^a=c_{AB}{}^C B_C{}^a-B_A{}^e B_B{}^f c_{ef}{}^a. \tag{8.14}$$

The first of these relations shows that the group G is now realized as a global group of diffeomorphisms $x^i\to x^i-\xi^i$, with

$$\xi^i=a^B B_B{}^i. \tag{8.15}$$

The relations (8.14) were given by Harnad and Pettitt[7] as the necessary conditions for the transformation law (8.6) [with M_B given by (8.11)] to represent a global group of diffeomorphisms of the form (8.15).

The procedure adopted in this section is the inverse of the usual one—we started with a gauge theory and "ungauged it," ending up with a global group of transformations. The Lagrangian density (5.15), if it exists, gives rise to a Lagrangian density

$$\mathscr{L}(\psi,\partial_i\psi,x^i)=\mathring{e}L(\psi,\mathring{D}_\alpha\psi), \tag{8.16}$$

where

$$\mathring{D}_\alpha\psi=\mathring{e}_\alpha{}^i(\partial_i\psi-\mathring{\Gamma}_i{}^a\overline{G}_a\psi), \tag{8.17}$$

for a theory that is covariant under a global group of diffeomorphisms (the $\mathring{\Gamma}_i{}^A$ are invariant specified functions of the coordinates, not fields). The Noether currents for this theory are defined by

$$\xi^i\mathscr{L}-\Pi^i\delta\psi=a^B J_B{}^i. \tag{8.18}$$

They are

$$J_B{}^j=B_B{}^i\hat{\theta}_i{}^j+B_B{}^A\theta_a{}^j, \tag{8.19}$$

where

$$\hat{\theta}_i{}^j=\mathscr{L}\delta_i{}^j-\Pi^j\partial_i\psi, \tag{8.20}$$

$$\theta_a{}^j=-\Pi^j\overline{G}_a\psi. \tag{8.21}$$

They satisfy

$$\partial_j J_B{}^j=0. \tag{8.22}$$

Alternatively, the Noether currents can be written in the form

$$J_B{}^j=E_B{}^A\theta_A{}^j, \tag{8.23}$$

where

$$\theta_\alpha{}^j=\mathscr{L}\mathring{e}_\alpha{}^j-\Pi^j\mathring{D}_\alpha\psi. \tag{8.24}$$

The conservation laws (8.22) then take the form

$$\partial_j\theta_A{}^j=\mathring{\Gamma}_j{}^B c_{AB}{}^C\theta_C{}^j, \tag{8.25}$$

which are of course the limiting cases of the source identities (5.11). The quantities $\theta_A{}^j$ are "intrinsic" currents for the group G and the $J_A{}^j$ are the "total" (intrinsic + orbital) currents.

The covariance of (8.16) is of course lost when the parameters a^A are made space-time dependent [which is tantamount to making the λ^A independent space-time-dependent functions by abandoning the constraint (8.2)]. The change in the Lagrangian is now

$$\delta\mathscr{L}=\partial_i(\xi^i\mathscr{L})-(\partial_i a^B)J_B{}^i. \tag{8.26}$$

Obviously, the covariance can be maintained by introducing auxiliary fields $\Gamma_i{}^A$ so as to revert to the original theory of Secs. V and VI.

Important particular cases of the foregoing theory arise when the π_α commute ($c_{\alpha\beta}{}^C=0$). The Poincaré group gauge theories and conformal gauge theory belong to this class. An appropriate choice for the $\mathring{\Gamma}_i{}^A$ in these cases is

$$\mathring{e}_i{}^\alpha=\delta_i{}^\alpha,\quad \mathring{\Gamma}_i{}^a=0. \tag{8.27}$$

The distinction between Latin and Greek indices, and the distinction between the generalized derivative and the ordinary partial derivative, now disappear. The constraint (8.2) on the transformation parameters is now

$$\partial_\gamma\lambda^A+\lambda^B c_{B\gamma}{}^A=0. \tag{8.28}$$

The matrix σ that solves (7.3) is

$$\sigma=e^{-\pi\cdot x},\quad \pi\cdot x=\pi_\alpha x^\alpha, \tag{8.29}$$

so that

$$E_B{}^A=(e^{-c\cdot x})_B{}^A,\quad c\cdot x=c_\alpha x^\alpha, \tag{8.30}$$

where the four matrices c_α are the adjoint representatives of the π_α,

$$(c_\alpha)_B{}^A=c_{B\alpha}{}^A. \tag{8.31}$$

On account of $c_{\alpha\gamma}{}^B=0$, we have

$$E_\alpha{}^\beta=\delta_\alpha{}^\beta,\quad E_\alpha{}^b=0, \tag{8.32}$$

and consequently

$$\xi^\alpha=a^\alpha+a^b E_b{}^\alpha \tag{8.33}$$

and

$$\delta\psi=a^\alpha\partial_\alpha\psi+a^b(E_b{}^\alpha\partial_\alpha+E_b{}^a\overline{G}_a)\psi. \tag{8.34}$$

The Noether currents in these cases are

$$J_\beta{}^j=\theta_\beta{}^j,\quad J_b{}^j=E_b{}^\alpha\theta_\alpha{}^j+E_b{}^a\theta_a{}^j. \tag{8.35}$$

The two pieces of the right-hand side of the final expression correspond to the "intrinsic" and "orbital" parts of the current.

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