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Gauging the conformal group

E A LORD* and P GOSWAMI†

*Department of Applied Mathematics, †Department of Physics, Indian Institute of Science, Bangalore 560012, India

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Abstract. It is demonstrated that Kibble's method of gauging the Poincaré group can be applied to the gauging of the conformal group. The action of the gauge transformations is the action of general spacetime diffeomorphisms (or coordinate transformations) combined with a local action of an 11-parameter subgroup of SO (4, 2). Because the translational subgroup is not an invariant subgroup of the conformal group the appropriate generalisation of the derivative of a physical field is not a covariant derivative in the usual sense, but this does not lead to any inconsistencies.

Keywords. Conformal group; gauge theories.

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1. Introduction

The concept of a gauge group has become a well-established feature of physical theories, of central importance. The standard method of gauging a non-abelian internal symmetry group G is due to Yang and Mills (1954). The group parameters are made spacetime-dependent and the covariance of the field equations is maintained by the introduction of auxiliary fields, the Yang–Mills potentials or gauge potentials (they are the components of a connection on a principal fiber bundle with spacetime as base space and G as fiber). The gauging of the Poincaré group by Kibble (1961) revealed that the gauging of groups that act on the points of spacetime as well as on the components of physical fields is a meaningful concept. The auxiliary fields in the case of the Poincaré group are essentially a tetrad and a metric-compatible but asymmetric linear connection on the spacetime. Taking the Lagrangian for the auxiliary fields to be the scalar curvature constructed from the tetrad and connection leads to a viable extension of Einstein's gravitational theory, now known as the ECKS (Einstein-Cartan-Kibble-Sciama) theory. An alternative Lagrangian, quadratic in curvature and torsion, also leads to a viable gravitational theory (von der Heyde 1976; Hehl 1980; Hehl *et al* 1980).

The Poincaré group is the group of isometries of Minkowski space. The two important candidates for extending the group are the affine group and the conformal group. Neither of these can be an exact symmetry of a realistic physical theory; they are to be considered as spontaneously broken symmetries. The gauging of the affine group has been carried out by Lord (1978); there are interesting indications that the affine gauge theory may be the correct extension of Poincaré gauge theory and could lead to an understanding of the relationship between the gravitational and the strong interactions (Hehl *et al* 1977, 1978). The conformal group is the fifteen-parameter group of diffeomorphisms on Minkowski space that preserves the light-cone structure. The action of this group on ordinary physical fields (representations of the Poincaré group) can be defined, and Lagrangian theories that are invariant under conformal transformations can be constructed (see for example Mack and Salam 1969). The breaking of the conformal symmetry is associated with particle masses and coupling constants that are not dimensionless.

Some general principles underlying the gauging of groups of spacetime diffeomorphisms have been worked out by Harnad and Pettitt (1976). The particular case of the conformal group was discussed by the same authors, in the language of fiber bundles and employing the concept of second order frames (Harnad and Pettitt 1977). The present work will demonstrate that the concept of second order frames is not necessary for the construction of gauge theories of the conformal group.

The approach to the gauging of the conformal group to be presented here is based on a straightforward generalisation of the method applied by Kibble to the Poincaré group.

2. The auxiliary fields

The conformal group contains an 11-parameter subgroup H, that leaves the origin ($x^\alpha = 0$) fixed. It is generated by $S_{\alpha\beta}$ (Lorentz rotations), Δ (dilatations) and κ_α (special conformal transformations) satisfying the commutation relations

$$\left.\begin{aligned} &[S_{\alpha\beta}, S_{\gamma\delta}] = \eta_{\beta\gamma} S_{\alpha\delta} - \eta_{\alpha\gamma} S_{\beta\delta} + \eta_{\alpha\delta} S_{\beta\gamma} - \eta_{\beta\delta} S_{\alpha\gamma}, \\ &[S_{\alpha\beta}, \Delta] = 0, \quad [S_{\alpha\beta}, \kappa_\gamma] = \kappa_\alpha \eta_{\beta\gamma} - \kappa_\beta \eta_{\alpha\gamma}, \\ &[\Delta, \kappa_\alpha] = \kappa_\alpha. \end{aligned}\right\} \tag{1}$$

Let ψ be a set of field components belonging to a finite-dimensional linear representation of H. The infinitesimal action $x^\alpha \to x^\alpha - \xi^\alpha$ of the conformal group on the points of Minkowski space is given by

$$\xi^\alpha = a^\alpha + x_\beta \omega^{\beta\alpha} + \rho x^\alpha + 2x^\alpha c \cdot x - c^\alpha x^2, \tag{2}$$

where a^α, $\omega^{\alpha\beta}$, ρ and c^α are constant parameters associated respectively with translation, Lorentz rotation, dilatation and special conformal transformation; the corresponding action on the field is

$$\left.\begin{aligned} &\delta\psi = \xi^i \partial_i \psi + \bar{\varepsilon}\psi, \\ &\bar{\varepsilon} = \frac{1}{2}\varepsilon^{\alpha\beta} S_{\alpha\beta} + \zeta\Delta + \zeta^\alpha \kappa_\alpha \end{aligned}\right\} \tag{3}$$

where $\delta\psi$ is the substantial variation ($\delta\psi = \psi'(x) - \psi(x)$) and

$$\left.\begin{aligned} &\zeta^\alpha = c^\alpha, \\ &\zeta = \rho + 2c \cdot x, \\ &\varepsilon^{\alpha\beta} = \omega^{\alpha\beta} + 2(x^\beta c^\alpha - x^\alpha c^\beta) \end{aligned}\right\} \tag{4}$$

(see, for example, Mack and Salam 1969).

The transformation law for the derivative of ψ is

$$\delta \partial_j \psi = \xi^i \partial_i \partial_j \psi + \partial_j \xi^i \partial_i \psi + \bar{\varepsilon} \partial_j \psi + \partial_j \bar{\varepsilon} \cdot \psi \tag{5}$$

(so long as we are discussing the global action of the conformal group in Minkowski space, we make no distinction between Greek and Latin indices). Now,

$$\partial_\gamma \xi^\alpha = \omega^\alpha_\gamma + \rho \delta^\alpha_\gamma + 2(c \cdot x \delta^\alpha_\gamma - c^\alpha x_\gamma - c_\gamma x^\alpha) = \varepsilon^\alpha_\gamma + \zeta \delta^\alpha_\gamma \tag{6}$$

and

$$\partial_\gamma \bar{\varepsilon} = 2(\zeta^\alpha S_{\alpha\gamma} + \zeta_\gamma \Delta). \tag{7}$$

so the transformation law for the derivative can be written in the form

$$\delta \partial_\gamma \psi = \xi^i \partial_i \partial_\gamma \psi + (\varepsilon^\alpha_\gamma + \zeta \delta^\alpha_\gamma) \partial_\alpha \psi + \bar{\varepsilon} \partial_\gamma \psi + 2(\zeta^\alpha S_{\alpha\gamma} + \zeta_\gamma \Delta) \psi. \tag{8}$$

We now gauge the group by allowing the parameters a^α, $\omega^{\alpha\beta}$, ρ and c^α to be spacetime-dependent. This is equivalent to allowing ξ^α $\varepsilon^{\alpha\beta}$, ζ and ζ^α to become independent of each other. The relations (6) and (7) no longer hold, so that the transformation law (5) of the derivative does not have the form (8). We now apply the usual Yang-Mills prescription: with the aid of auxiliary fields, we can construct a *generalised derivative* ψ_γ which transforms with a transformation law like (8) *even though the parameters are spacetime-dependent*,

$$\delta \psi_\gamma = \xi^i \partial_i \psi_\gamma + (\varepsilon^\alpha_\gamma + \zeta \delta^\alpha_\gamma) \psi_\alpha + \bar{\varepsilon} \psi_\gamma + 2(\zeta^\alpha S_{\alpha\gamma} + \zeta_\gamma \Delta) \psi. \tag{9}$$

The generalised derivative is constructed from ψ, $\partial_j \psi$ and auxiliary fields e^j_γ and $\bar{\Gamma}_j$ according to

$$\psi_\gamma = e^j_\gamma \psi_j, \quad \psi_j = \partial_j \psi - \bar{\Gamma}_j \psi \tag{10}$$

(the $\bar{\Gamma}_j$ are linear combinations of the generators of H). The necessary transformation laws of e^j_α and $\bar{\Gamma}_j$, in order for (3) and (5) to lead to the transformation law (9) for ψ_γ, are uniquely determined. We have

$$\delta \psi_\gamma = \delta e^j_\gamma \psi_j - e^j_\gamma \delta \bar{\Gamma}_j \psi + e^j_\gamma \delta \partial_j \psi - e^j_\gamma \bar{\Gamma}_j \delta \psi$$

which leads to

$$\begin{aligned} \delta e^j_\gamma \psi_j - e^j_\gamma \delta \bar{\Gamma}_j \psi = {} & [\xi^i \partial_i e^j_\gamma - e^i_\gamma \partial_i \xi^j + (\varepsilon^\alpha_\gamma + \zeta \delta^\alpha_\gamma) e^j_\alpha] \psi_j \\ & - e^j_\gamma [\xi^i \partial_i \bar{\Gamma}_j + \bar{\Gamma}_i \partial_j \xi^i + \partial_j \bar{\varepsilon} + [\bar{\varepsilon}, \bar{\Gamma}_j]] \psi \\ & + 2(\zeta^\alpha S_{\alpha\gamma} + \zeta_\gamma \Delta) \psi. \end{aligned}$$

The required expressions for δe^j_γ and $e^j_\gamma \delta \bar{\Gamma}_j$ are given by picking out the coefficients of ψ_j and ψ. It is natural to regard the fields e^j_γ as the components of a tetrad. Assuming the matrix (e^j_γ) to be nonsingular, with inverse (e^γ_j), we have

$$\delta e^\gamma_j = \xi^i \partial_i e^\gamma_j + e^\gamma_i \partial_j \xi^i - e^\alpha_j (\varepsilon^\gamma_\alpha + \zeta \delta^\gamma_\alpha), \tag{11}$$

$$\delta \bar{\Gamma}_j = \xi^i \partial_i \bar{\Gamma}_j + \bar{\Gamma}_i \partial_j \xi^i + \partial_j \bar{\varepsilon} + [\bar{\varepsilon}, \bar{\Gamma}_j] - 2 e^\gamma_j (\zeta^\alpha S_{\alpha\gamma} + \zeta_\gamma \Delta). \tag{12}$$

Observe that the action (3) of the gauged conformal group can be interpreted as the combined action of a general coordinate transformation (GCT) and an 'internal' gauge group H. The auxiliary fields e^γ_j and $\bar{\Gamma}_j$ transform like covariant vectors under the GCT. Under the gauge group H, the tetrad is rotated and dilated. The final term in (12) shows that $\bar{\Gamma}_j$ is *not* the connection for the gauge group H. Indeed, it was already apparent from the presence of the final term in (9) that ψ_γ is not a *covariant* derivative associated with the group H, in the usual sense—its transformation law is linear but inhomogeneous. This is a special peculiarity of the conformal gauge theory that is not shared by the Poincaré gauge theory. It can be traced to the fact that the translations do not form an *invariant* subgroup of the conformal group. In the following section we shall see how fully covariant Lagrangian theories can be constructed with the aid of the new derivative ψ_γ, in spite of the fact that this derivative does not have a homogeneous transformation law.

The transformation laws (11) and (12) can be better understood as follows. Consider a purely *internal* SO(4,2) symmetry generated by π_α, $S_{\alpha\beta}$, Δ and κ_α satisfying (1) together with

$$\left.\begin{aligned} &[\pi_\alpha, \pi_\beta] = 0, \quad [\pi_\alpha, S_{\beta\gamma}] = \eta_{\alpha\beta}\pi_\gamma - \eta_{\alpha\gamma}\pi_\beta, \\ &[\pi_\alpha, \Delta] = \pi_\alpha, \quad [\pi_\alpha, \kappa_\beta] = 2(\eta_{\alpha\beta}\Delta + S_{\beta\alpha}). \end{aligned}\right\} \tag{13}$$

and consider the transformation law of the connection

$$\Gamma_j = e^\alpha_j \pi_\alpha + \bar{\Gamma}_j \tag{14}$$

under the simultaneous action of a GCT and H:

$$\delta\Gamma_j = \xi^i\partial_i\Gamma_j + \Gamma_i\partial_j\xi^i + \partial_j\bar{\varepsilon} + [\bar{\varepsilon}, \bar{\Gamma}_j]. \tag{15}$$

We find precisely the transformation laws (11) and (12) for the two parts of Γ_j. Thus the tetrad and the 'pseudo-connection' $\bar{\Gamma}_j$ together constitute a connection for the group SO (4, 2).

3. Lagrangian theories

Let $L(\psi, \partial_\gamma\psi)$ be a Lagrangian for a theory that is invariant under the global conformal group. That is,

$$L = \frac{\partial L}{\partial\psi}\delta\psi + \Pi^i\delta\partial_i\psi = \partial_i(\xi^i L), \tag{16}$$

where

$$\Pi^i = \partial L/\partial\partial_i\psi. \tag{17}$$

Employing the field equations

$$\frac{\delta L}{\delta\psi} = \frac{\partial L}{\partial\psi} - \partial_i\Pi^i = 0, \tag{18}$$

we get the Noether identities in the form

$$\partial_i(\xi^i L - \Pi^i\delta\psi) = 0. \tag{19}$$

The Noether currents are energy-momentum, angular momentum, the dilatation current and the special conformal current, defined as the ceofficients of the parameters in the expression

$$\xi^i L - \Pi^i\delta\psi = a^\alpha\theta^i_\alpha + \frac{1}{2}\omega^{\alpha\beta}\mathscr{M}^i_{\alpha\beta} + \rho\mathscr{D}^i + c^\alpha\mathscr{K}^i_\alpha. \tag{20}$$

They are, explicitly,

$$\left.\begin{aligned} \theta^i_\alpha &= L\delta^i_\alpha - \Pi^i \partial_\alpha \psi, \\ \mathcal{M}^i_{\alpha\beta} &= x_\alpha \theta^i_\beta - x_\beta \theta^i_\alpha + \tau^i_{\alpha\beta}, \\ \mathcal{D}^i &= x^\alpha \theta^i_\alpha + \Delta^i, \\ \mathcal{K}^i_\alpha &= (2x^\beta x_\alpha - x^2 \delta_\alpha \beta)\,\theta^i_\beta + 2(x^\beta \tau^i_{\alpha\beta} + x_\alpha \Delta^i) + \kappa^i_\alpha \end{aligned}\right\} \tag{21}$$

(c.f. Mack and Salam 1969), where the *intrinsic* currents are

$$\left.\begin{aligned} \tau^i_{\alpha\beta} &= -\Pi^i S_{\alpha\beta}\psi, \\ \Delta^i &= -\Pi^i \Delta\psi, \\ \kappa^i_\alpha &= -\Pi^i \kappa_\alpha \psi. \end{aligned}\right\} \tag{22}$$

Alternatively, observe that (16) is

$$\frac{\partial L}{\partial \psi}\,\delta\psi + \Pi^i \delta\,\partial_i \psi = \xi^i \partial_i L + 4\zeta\, L. \tag{23}$$

Substituting (3) and (8) into this expression gives

$$\frac{\partial L}{\partial \psi}\,\bar{\varepsilon}\psi + \Pi^\gamma\left[\bar{\varepsilon}\partial_\gamma\psi + (\varepsilon^\alpha_\gamma + \zeta\delta^\alpha_\gamma)\,\partial_\alpha\psi + 2\,(\zeta_\alpha\Delta + \zeta^\alpha S_{\alpha\gamma})\,\psi\right] = 4\zeta\, L. \tag{24}$$

Equating coefficients of $\omega^{\alpha\beta}$, ρ and c^α gives the following conditions for the field equations to be conformally invariant:

$$\left.\begin{aligned} &\frac{\partial L}{\partial \psi}\, S_{\alpha\beta}\psi + \Pi^\gamma\,(S_{\alpha\beta}\partial_\gamma\psi + \eta_{\alpha\gamma}\,\partial_\beta\psi - \eta_{\alpha\beta}\,\partial_\gamma\psi) = 0, \\ &\frac{\partial L}{\partial \psi}\,\Delta\psi + \Pi^\gamma\,(1+\Delta)\,\partial_\gamma\psi = 4L, \\ &\frac{\partial L}{\partial \psi}\,\kappa_\alpha\psi + \Pi^\gamma\left[\kappa_\alpha\partial_\gamma\psi + 2(\eta_{\alpha\gamma}\Delta + S_{\alpha\gamma})\,\psi\right] = 0. \end{aligned}\right\} \tag{25}$$

When the parameters of the conformal group are made spacetime-dependent, the covariance of the theory is lost. In fact, the change in the Lagrangian L, under the action of the group with spacetime-dependent parameters, is given by

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial \psi}\,\delta\psi + \Pi^i \delta\,\partial_i \psi = \partial_i\,(\Pi^i \delta\psi) \\ &= \partial_i\,(\xi^i L) - \partial_i\,(\xi^i L - \Pi^i \delta\psi). \end{aligned}$$

That is,

$$\delta L = \partial_i\,(\xi^i L) - (\partial_i a^\alpha)\,\theta^i_\alpha - \frac{1}{2}\,(\partial_i \omega^{\alpha\beta})\mathcal{M}^i_{\alpha\beta} - (\partial_i \rho)\,\mathcal{D}^i - (\partial_i c^\alpha)\mathcal{K}^i_\alpha. \tag{26}$$

This expression is analogous to the expression given by Mukunda (1982) for the Poincaré group. We can attempt to restore the covariance of the theory by replacing the derivative of ψ by the generalised derivative ψ_γ. Since ψ_γ was contrived to have the same transformation law under the local action that $\partial_\gamma\psi$ had under the global action, it follows that (24) and (25) will hold for the modified Lagrangian $L(\psi, \psi_\gamma)$, if $\partial_\gamma \psi$ is replaced throughout by ψ_γ. Hence, for the new Lagrangian,

$$\delta L = \frac{\partial L}{\partial \psi}\,\delta\psi + \frac{\partial L}{\partial \psi_\gamma}\,\delta\psi_\gamma = \xi^i \partial_i L + 4\zeta L \tag{27}$$

(c.f. (23)). The right side is now no longer equal to $\partial_i(\xi^i L)$. A further modification of the Lagrangian is required. We introduce an auxiliary field e and define $\mathscr{L} = eL$ so that

$$\delta \mathscr{L} = \partial_i(\xi^i \mathscr{L}). \tag{28}$$

The required transformation law for e is

$$\delta e = \partial_i(\xi^i e) - 4\zeta e. \tag{29}$$

That is, e is a scalar density under a GCT and undergoes dilation under the local action H. An obvious prescription for e is the determinant of the tetrad,

$$e = |e_i^\alpha|. \tag{30}$$

Thus, as for the Poincaré group (and for internal symmetry groups), there is a simple prescription for converting a Lagrangian theory, invariant under the global conformal group, to a theory invariant under the conformal gauge group (i.e. under GCT and the local action of H): replace derivatives by the generalised derivatives (10) and multiply by the tetrad determinant (30). Then add on a Lagrangian for the auxiliary fields—the obvious choice is

$$\sqrt{-g}\, g^{ij} g^{kl}\, \mathrm{trace}\, F_{ik} F_{jl}, \tag{31}$$

where

$$g_{ij} = e_i^\alpha e_j^\beta \eta_{\alpha\beta}, \tag{32}$$

and

$$F_{ij} = \partial_i \Gamma_j - \partial_j \Gamma_i - [\Gamma_i, \Gamma_j], \tag{33}$$

constructed from (14) with π_α, $S_{\alpha\beta}$, etc. belonging to the adjoint representation of SO (4,2).

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