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Metric-Affine Variational Principles in General Relativity
II. Relaxation of the Riemannian Constraint

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Abstract

We continue our investigation of a variational principle for general relativity in which the metric tensor and the (asymmetric) linear connection are varied independently. As in Part I, the matter Lagrangian is minimally coupled to the connection and the gravitational Lagrangian is taken to be the curvature scalar, but we now relax the Riemannian constraint as far as possible—that is, as far as the projective invariance of the assumed gravitational Lagrangian will allow. The outcome of this procedure is a gravitational theory formulated in a volume-preserving space-time (i.e., with torsion and tracefree nonmetricity). The vanishing of the trace of the nonmetricity is due to the remaining vector constraint. We also discuss the physical significance of the relaxation of the Riemannian constraint, the possible relaxation of the vector constraint, the notion of the hypermomentum current, and its possible relation to elementary particle physics.

§(1): *Introduction*

In Part I: Riemannian Spacetime [1],[3] we analyzed the variational principle for general relativity (GR), in which the components to be varied were those of the metric and an asymmetric linear connection. The Riemannian structure of the space-time was imposed by the introduction of Lagrange multipliers. These

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[3] We shall refer to this paper hereafter simply as I; our standard notation is that of Schouten [2] and is summarized in I.

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multipliers turned out to be the components of the *hypermomentum current*, which is thus related to the "*constraint force*" that imposes the Riemannian structure. In Part II, we now adopt the approach of Hamel [3] : If a constraint is removed, the corresponding constraint force becomes an *intrinsic force* depending primarily on the deformations that were previously forbidden by the constraint. A constraint in physics is usually not completely rigid; the notion of rigidity appears only within the domain of some approximation. Indeed, rigid space-time structures are, as argued by Einstein ([4], p. 36), contrary to the spirit of GR.

However, if the gravitational Lagrangian contains *hidden invariances*, the complete relaxation of all constraints cannot be carried out; one encounters residual constraints associated with the hidden invariance. This is the case with the curvature scalar density $\mathcal{R}$ adopted as the gravitational Lagrangian in Part I. The projective invariance of $\mathcal{R}$ implies that it can respond only to 60 of the 64 degrees of freedom associated with the linear connection. A matter Lagrangian obtained by minimally coupling the matter to the connection will not in general be projectively invariant. Thus, in general, the *whole* of the Riemannian constraint cannot be relaxed as long as $\mathcal{R}$ is taken to be the whole of the gravitational part of the Lagrangian; a residual vector restraint will have to remain. Theories with such a residual restraint will be seen, in Section 8, to be equivalent to theories without constraint whose matter Lagrangians are contrived, by violation of minimal coupling, to be projectively invariant. In order to be able to fully relax the Riemannian constraint, without violating minimal coupling, and thus to arrive at the full metric-affine (L_4, g) theory [5], it is therefore necessary to generalize the gravitational part of the Lagrangian, so as to break its projective invariance. This problem will be discussed in the conclusion to this paper.

From the point of view of differential geometry, the metric properties of a space, and its affine properties, i.e., those related to parallel transport, are conceptually distinct. Yet, in GR, the metric properties of space-time are regarded as fundamental, whereas the affine properties are defined in terms of them and have only a subsidiary role; the space-time of physics is assumed a priori to be Riemannian. Now, in I, it was established that the Riemannian structure of space-time in GR can consistently be regarded as due to the action of a constraint force. From this viewpoint, the Riemannian assumption appears fairly artificial. In this paper, the arbitrary imposition of Riemannian structure is replaced by a situation in which any lack of independence between metric and connection would arise dynamically as a consequence of the nature of the coupling between the connection and the matter fields that give rise to the space-time geometry. The nature of this coupling is determined by a minimal coupling hypothesis (cf. [5, 6]).

In Section 2, the properties of the curvature scalar density $\mathcal{R}$ are investigated, with particular emphasis on its projective invariance, and a volume-preserving linear connection is defined. In Section 3, the Bianchi identities for torsion and curvature are spelled out. Their contracted versions are brought into a projectively invariant form. In Section 4, the matter Lagrangian with minimal coupling to the connection is introduced and the important relations between energy-momentum and hypermomentum are derived. Section 5 introduces the part of the Lagrangian associated with the constraints. The Einstein-Cartan-Sciama-Kibble theory [7] (EC theory) is shown to arise from a *partial* relaxation of the constraints, in Section 6. In Section 7 we illustrate the way in which inconsistencies arise when a relaxation is attempted which is forbidden by the projective invariance. In Section 8 all the constraints which the projectively invariant $\mathcal{R}$ will allow to be relaxed, are relaxed, and we are left with a single vector constraint. The physical implications of the approach are discussed in Section 9.

The literature on metric-affine variational principles we collected already in I. We emphasize again the usefulness of the formalism of Kopczyński [8] and of Trautman [9] for our purposes. In the meantime a coherent framework for an affine gauge theory has been given by Lord [5], whereas Hennig and Nitsch [10] and Norris, Fulp, and Davis [11] supplied a corresponding differential geometrical analysis. Interesting work on metric-affine theory has also been recently done by Davis [12, 13], Goenner [14] (who cleared up a misunderstanding of Tsamparlis [15], inter alia), by Szczyrba [16, 17], Yasskin [18], and, for a complex metric and a complex connection, by Kunstatter [19]. For somewhat related work see also Stachel [20]. Further work will be cited at the appropriate place in our discussion.

§(2): *The Curvature Scalar Density and the "Dagger Connection"*

The gravitational field is represented by the 10 components of the symmetric metric tensor g_{ij} and the 64 components of the asymmetric linear connection $\Gamma_{ij}{}^{k}$ (or, equivalently, the 24 *torsion* components $S_{ij}{}^{k} := \Gamma_{[ij]}{}^{k}$ plus the 40 *nonmetricity* components $Q_{ijk} = -\overset{\Gamma}{\nabla}_i g_{jk}$). As in I, we take the gravitational Lagrangian density to be

$$\frac{1}{2\kappa}\mathcal{R}(g,\Gamma) := \frac{1}{2\kappa}\, e g^{ij} R_{ij}(\Gamma) \tag{1}$$

We propose to study the sequence of theories arising from the Lagrangian density

$$\frac{1}{2\kappa}\mathcal{R}(g,\Gamma) + \mathcal{L}(g,\Psi,\overset{\Gamma}{\nabla}\Psi) + \mathcal{C} \tag{2}$$

when the constraints are successively relaxed. $\mathcal{L}$ is the matter Lagrangian in which the coupling to the connection is specified by a minimal coupling hypothesis. $\mathcal{C}$ is the term which contains the constraint forces (Lagrange multipliers).

We wish to emphasize, however, that we are adopting this *ansatz* mainly because of its simplicity. It serves as a means of illustrating the *general principles* involved in the idea of relaxation of constraints. We do not regard it as an acceptable candidate for the gravitational part of the Lagrangian of a fully developed metric-affine theory (see the discussion in Section 9). In our view, its projective invariance is a defect which rules out this possibility. But it is precisely this defect which, in the present context, renders it useful as a means of investigating the way in which hidden invariances in a Lagrangian hinder the process of relaxation of constraints. Moreover, since (1) is the "obvious" metric-affine generalization of the gravitational Lagrangian of GR, its investigation gives some insight into the deeper structural aspects of that theory.

The symmetric part of the generalized Einstein tensor G^{ij} is the functional derivative of $\mathcal{R}$ with respect to the metric, and the functional derivative of $\mathcal{R}$ with respect to the connection is the *Palatini tensor* $P_k{}^{ji}$:

$$\delta\mathcal{R} \sim e(-G^{(ij)}\delta g_{ij} + 2P_k{}^{ji}\delta\Gamma_{ij}{}^k) \tag{3}$$

Here $\sim$ denotes equality apart from an irrelevant divergence.

The Lagrangian (1) is *projectively invariant:*

$$\Gamma_{ij}{}^k \longrightarrow \Gamma_{ij}{}^k + \lambda_i\delta_j^k \tag{4a}$$

$$g_{ij} \longrightarrow g_{ij} \tag{4b}$$

That is, it is invariant under certain changes in the connection, whereas the metric is kept fixed.[4] Geometrically, the meaning of this transformation is as follows: A linear connection specifies a definition of the *parallel transport* of a vector along a curve. Two alternative definitions of parallel transport differ only in the change they produce in the length of the vector, if and only if the associated connections are related by a transformation of the form (4) (Schouten [2]). In the context of this paper, the important consequence of the projective invariance of $\mathcal{R}$ is that the Lagrangian (1) is capable of responding to only 60 of the 64 degrees of freedom associated with the affine connection.

In order to gain a better understanding of the nature of projective invariance, consider the splitting of the connection into a Riemannian part and a non-Riemannian part

$$\Gamma_{ij}{}^k = \left\{ {k \atop ij} \right\} + W_{ij\cdot}{}^k \tag{5}$$

where

$$W_{ij\cdot}{}^k = S_{ij\cdot}{}^k - S_{j\cdot i}^{k} + S^k{}_{\cdot ij} + \tfrac{1}{2}\,(Q_{ij\cdot}{}^k + Q_{j\cdot i}^{k} - Q^k{}_{\cdot ij}) \tag{6}$$

[4] Other types of "projective" transformations have been introduced which are based on a combination of (4a) with certain related transformations of the tetrad (see, for example, Davis [13]) or of the metric (see, for example, Smalley [21]). Throughout the present work, "projective invariance" will mean simply invariance under (4).

Correspondingly, a vector or tensor, parallely transported from x^i to $x^i + dx^i$, will undergo a change of components due to the purely metric properties of the space-time and the choice of the coordinate system (specified by the Riemannian definition of parallel transport based on the Christoffel symbols), and in addition it will be changed by a linear transformation $\alpha_j^{\ k} := W_{ij.}^{\ \ k}\, dx^i$. The "deformation" α can be split into a *rotation*, a *shear*, and a *dilation*, corresponding to the decomposition of the tensor $W_{ij.}^{\ \ k}$:

$$W_{ijk} = + W_{i[jk]} + (W_{i(jk)} - \tfrac{1}{4} g_{jk} W_{il.}^{\ \ l}) + \tfrac{1}{4} g_{jk} W_{il.}^{\ \ l} \tag{7}$$

The *contortion* $M_{ijk} := - W_{i[jk]}$ is, in this context, more fundamental than the torsion in that it has a distinct geometrical meaning with regard to the rotational part of the deformation produced by parallel transport, whereas the torsion $S_{ij.}^{\ \ k} = W_{[ij].}^{\ \ \ k}$ involves a skew-symmetrization on the "wrong" index pair and thus has no such simple geometrical interpretation. The change in the length of a vector is produced by the trace of $W_{ij.}^{\ \ k}$:

$$W_{il.}^{\ \ l} = 2Q_i := \tfrac{1}{2} Q_{il.}^{\ \ l} \tag{8}$$

The vector Q_i is the *Weyl vector.*

The connection can now be decomposed into a part which preserves the length of a vector undergoing parallel transport, and a part that changes the length:

$$\Gamma_{ij}^{\ \ k} = {}^{\dagger}\Gamma_{ij}^{\ \ k} + \tfrac{1}{2} Q_i \delta_j^k \tag{9}$$

The "dagger connection" defined by this expression will be called the *volume-preserving* part of the connection $\Gamma_{ij}^{\ \ k}$, for the following reason: The law of parallel transport of the volume measure $e := (-g)^{1/2}$ is

$$\delta e = \Gamma_{il.}^{\ \ l} e\, dx^i = e(\partial_i \ln e + 2Q_i)\, dx^i \tag{10}$$

(Schouten [2]). The first term in the parentheses is a "metrical dilation" produced by the Riemannian (Christoffel symbol) part of the connection, because e is a density, and the second term is an "affine dilation" produced by the non-Riemannian aspects of the connection. The affine dilation is zero if and only if the Weyl vector vanishes

$$\overset{\Gamma}{\nabla}_i e = -2eQ_i = 0 \tag{11}$$

and a connection with this property is said to be volume preserving[5] with respect to the metric g_{ij}. Every connection is related by a projective transformation to a volume-preserving connection, and this is uniquely the "dagger connection" defined by (9). Note that the dagger connection is projectively invari-

[5] Schouten's [2] definition of volume preserving (teleparallelism for scalar densities) is slightly more general than ours (no affine dilation).

ant (i.e., invariant under projective transformations applied to the original connection) and has vanishing Weyl vector:

$$^{\dagger}\nabla_i e = -2e\,^{\dagger}Q_i = 0 \tag{12}$$

The projective transformation law for the Riemann tensor is

$$R_{ijk.}{}^{l} \longrightarrow R_{ijk.}{}^{l} + 2\partial_{[i}\lambda_{j]}\delta^l_k \tag{13}$$

and hence the Riemann tensor of the volume preserving connection is

$$^{\dagger}R_{ijk.}{}^{l} \equiv R_{ijk.}{}^{l} - \partial_{[i}Q_{j]}\delta^l_k \tag{14}$$

In a metric-affine space, there are three possible contractions of the curvature tensor, namely,

$$R_{jk} := R_{ljk.}{}^{l}, \qquad R^{l}_{.jl.}{}^{k} := g^{il}R_{ijl.}{}^{k}, \qquad V_{ij} := R_{ijl.}{}^{l} \tag{15}$$

The trace of the first is R, that of the second $-R$, and the third being skew-symmetric, is traceless. Thus R is unambiguously defined. Its projective invariance follows immediately from (13), and can be expressed in the form

$$R(\Gamma) \equiv R(^{\dagger}\Gamma) \tag{16}$$

Thus, the part of the connection missing from R is identified as the Weyl vector Q_i. The form of equation (14) leads to the conclusion that, for a Riemann tensor to be projectively invariant, it is not necessary for the Weyl vector to vanish; it is sufficient if $V_{ij} = 4\partial_{[i}Q_{j]} = 0$. But in order to have a volume-preserving space-time, it is, according to (11), necessary for Q_i to vanish.

From the projective invariance of $\mathcal{R}$, it follows immediately that the Palatini tensor in (3) is projectively invariant,

$$P_{k}^{.ji} \equiv {}^{\dagger}P_{k}^{.ji} \tag{17}$$

and that it is traceless:

$$P_{l}^{\;lk} = 0 \tag{18}$$

Because of (17), the expression for the Palatini tensor in terms of the torsion and nonmetricity can be slightly simplified by writing it in terms of the torsion and nonmetricity of the volume-preserving connection. We get [cf. I, equations (4.20), (4.21)]

$$P_{ij.}{}^{k} = {}^{\dagger}T_{ij.}{}^{k} + \delta_{[i}^{\;k}\,{}^{\dagger}Q_{l]j.}{}^{l} \tag{19}$$

where

$$^{\dagger}T_{ij.}{}^{k} = {}^{\dagger}S_{ij.}{}^{k} + 2\delta_{[i}^{\;k}\,{}^{\dagger}S_{j]l.}{}^{l} \tag{20}$$

is the modified torsion tensor of the volume-preserving connection.

For completeness and ease of reference, the relations between the Palatini tensor, and the contortion and nonmetricity, obtained in I, will be restated.

They are

$$-P_{(ij).}{}^{k} = \tfrac{1}{2}\left({}^{\dagger}Q_{(ij).}{}^{k} - \delta_{(i}^{k}\,{}^{\dagger}Q^{l}_{.j)l}\right) \tag{21a}$$

$$-P_{[ij].}{}^{k} = M_{[ij].}{}^{k} - \delta_{[i}^{k}M^{l}_{.j]l} \tag{21b}$$

where M_{ijk} is the contortion tensor,

$$\begin{aligned} M_{ijk} &= -S_{ijk} + S_{jki} - S_{kij} - Q_{[jk]i} \\ &= -{}^{\dagger}S_{ijk} + {}^{\dagger}S_{jki} - {}^{\dagger}S_{kij} - {}^{\dagger}Q_{[jk]i} \end{aligned} \tag{22}$$

Inverting the equations (21) gives

$$-M_{kij} = P_{[ij]k} + P_{[jk]i} + P_{[ki]j} - \tfrac{1}{2}g_{k[i}P_{j]l.}{}^{l} = {}^{\dagger}W_{k[ij]} \tag{23}$$

$$\tfrac{1}{2}\,{}^{\dagger}Q_{kij} = P_{(ij)k} - P_{(jk)i} - P_{(ki)j} + \tfrac{1}{2}g_{ij}P_{(kl).}{}^{l} = {}^{\dagger}W_{k(ij)} \tag{24}$$

From (21) and (23) we conclude that the contortion vanishes if, and only if, $P_{[ij]k} = 0$, and the traceless part of the nonmetricity vanishes if, and only if, $P_{(ij)k} = 0$. These equations also allow us, after some algebra, to express the Palatini tensor directly in ${}^{\dagger}W$:

$$P_{ij}{}^{k} = \tfrac{1}{2}\left(-{}^{\dagger}W_{ji}{}^{k} + \delta_{j}^{k}\,{}^{\dagger}W^{l}_{.il} - {}^{\dagger}W_{i.j}^{.k} + \delta_{i}^{k}\,{}^{\dagger}W^{l}_{.lj}\right) \tag{25}$$

One can read off part I, equation (4.24), directly from (25).

Finally, we give an expression for the decomposition of the Lagrangian density $\mathfrak{R}$ into a Riemannian part (the Riemannian curvature scalar density) and a non-Riemannian part:

$$\mathfrak{R}(\Gamma) \sim \mathfrak{R}(\{\ \}) + e\,{}^{\dagger}P_{k}^{\cdot ji}\,{}^{\dagger}W_{ij.}{}^{k} \tag{26}$$

This result can be established as follows: When the connection in the definition of the curvature scalar is expanded according to (5), the resulting expression has the overall structure

$$\mathfrak{R}(\Gamma) = \mathfrak{R}(\{\ \}) + \partial_i\mathfrak{N}^{i}(W) + \mathfrak{N}(W) \tag{27}$$

where $\mathfrak{N}^{i}$ is a vector density, homogenous linear in the components of W, and $\mathfrak{N}$ is a scalar density, *homogenous quadratic* in the components of W. (This result is most readily established in a Riemannian coordinate system, so that $\{^{k}_{ij}\} = 0$ at some point.) Now, the divergence term does not contribute to a functional derivative of $\mathfrak{R}$, so that

$$eP_{k}^{\cdot ji} := \tfrac{1}{2}\,\delta\mathfrak{R}/\delta\Gamma_{ij}{}^{k} = \tfrac{1}{2}\,\delta\mathfrak{R}/\delta W_{ij.}{}^{k} = \tfrac{1}{2}\,\partial\mathfrak{N}/\partial W_{ij.}{}^{k} \tag{28}$$

Therefore

$$eW_{ij.}{}^{k}\,P_{k}^{\cdot ji} = \tfrac{1}{2}\,W_{ij.}{}^{k}\,\partial\mathfrak{N}/\partial W_{ij.}{}^{k} = \mathfrak{N} \tag{29}$$

(the last step follows from Euler's theorem). The result, equation (26), is therefore established, since ${}^{\dagger}W_{ij}{}^{k}$ is just the traceless part of $W_{ij.}{}^{k}$. Incidentally, since $\mathcal{R}(\Gamma) \equiv \mathcal{R}({}^\dagger\Gamma)$, we could have carried through the proof ab initio in terms of ${}^\dagger W$.

§(3): *The Bianchi Identities for Torsion and Curvature*

In a metric-affine space-time, the Bianchi identity for the torsion (called "second identity for the curvature" in Schouten [2])

$$R_{[ijk]}{}^{l} \equiv 2\overset{\Gamma}{\nabla}_{[i}S_{jk]}{}^{l} - 4S_{[ij}{}^{m}S_{k]m}{}^{l} \tag{30}$$

and the ("third") identity

$$R_{li(kj)} \equiv \overset{\Gamma}{\nabla}_{[l}Q_{i]kj} + S_{li}{}^{m}Q_{mkj} \tag{31}$$

lead to an important relation for the divergence of the Palatini tensor. We contract (30), remember the definition (15), and find

$$R_{[ij]} \equiv \hat{\nabla}_k T_{ij.}{}^{k} - \tfrac{1}{2} V_{ij} \equiv \hat{\nabla}_k T_{ij.}{}^{k} - \partial_{[i}\Gamma_{j]k}{}^{k} \tag{32}$$

where $\hat{\nabla}_k := \overset{\Gamma}{\nabla}_k + 2S_{kl.}{}^{l}$ represents the covariant divergence operator and $T_{ij.}{}^{k} := S_{ij.}{}^{k} + 2\delta_{[i}^{k}S_{j]l.}{}^{l}$ the modified torsion tensor. Observe that in a volume-preserving space-time we simply have $R_{[ij]} = \hat{\nabla}_k T_{ij.}{}^{k}$. On rearranging (32), we get

$$\begin{aligned} eR_{[ij]} &\equiv \hat{\nabla}_k[e(T_{ij.}{}^{k} - 2\delta_{[i}^{k}Q_{j]})] \\ &\equiv \hat{\nabla}_k[e(P_{ij.}{}^{k} + \tfrac{1}{2}Q_{ij.}{}^{k} - \tfrac{1}{2}\delta_i^k Q_{lj.}{}^{l})] \end{aligned} \tag{33}$$

We subtract the contraction of (31) (multiplied by e) from (33) and obtain, after some heavy algebra,

$$\begin{aligned} \hat{\nabla}_k(eP_{i}{}^{jk}) &\equiv eg_{ik}(R^{[kj]} - R_{l}{}^{k(jl)}) \\ &\equiv eg_{ik}({}^{*}R^{*jk} - G^{(jk)}) \end{aligned} \tag{34}$$

The transformation to the second line will be discussed further down. It is not difficult to show that both sides of (34) are separately projectively invariant.

The Bianchi identity for the curvature reads (see [2])

$$\overset{\Gamma}{\nabla}_{[i}R_{jk]l.}{}^{m} \equiv 2S_{[ij.}{}^{n}R_{k]nl.}{}^{m} \tag{35}$$

On twofold contraction, we can derive a divergence relation for an Einstein-type tensor. Of course, like the divergence relation (34) for the Palatini tensor, we would like to obtain a projectively invariant form of such a relation. We have to note, however, that the naive generalization of the *Einstein tensor*

$$G_i{}^{j} := R_i{}^{j} - \tfrac{1}{2}\delta_i^j R \tag{36}$$

is not projectively invariant in a metric-affine space-time, even though its symmetric part $G_{(ij)}$ is, as is readily apparent from (13). Thus let us look for a projectively invariant second-rank tensor which, upon specialization to a U_4, reduces to the Einstein tensor (36).

In this context two observations are valuable. The right *dual* of the curvature tensor

$$R^*{}_{klab} := \tfrac{1}{2} R_{kl}{}^{[cd]} \epsilon_{abcd} \tag{37}$$

where the ϵ symbols obey (cf. [22])

$$\epsilon^{ijkl}\epsilon_{abcd} = -4!\,\delta_{[a}^{\,i}\,\delta_b^j \delta_c^k \delta_{d]}^l =: -\delta_{abcd}^{ijkl} \tag{38}$$

is clearly projectively invariant, as a comparison with (13) shows. Furthermore, the contracted *double dual* of the curvature tensor in GR coincides with the Einstein tensor. Thence, we derive the double dual

$$*R*^{ij}_{\cdot\cdot ab} := \tfrac{1}{4}\, \epsilon^{ijkl} R_{kl}^{\cdot\cdot}{}^{[cd]} \epsilon_{abcd} \tag{39}$$

the contraction of which

$$*R*^{i}_{\cdot j} := *R*^{ki}_{\cdot\cdot jk} = \tfrac{1}{2}\,(R_j^{\cdot i} - R_{kj}^{\cdot\cdot ki} - \delta_j^i R) \tag{40}$$

indeed reduces to the Einstein tensor $G_i^{\cdot j}$ (36) in a U_4.

An explicit form of (39) reads

$$*R*^{ij}_{\cdot\cdot ab} = -R_{ab}^{\cdot\cdot[ij]} + 2R_{k[a}^{\cdot\cdot\, k[i}\delta_{b]}^{j]} - 2R_{[a}^{[i}\delta_{b]}^{j]} + R\delta_{a}^{[i}\delta_{b}^{j]} \tag{41}$$

Observe that the twice double dual of $R_{kl}^{\cdot\cdot cd}$ gives

$$*(*R*)*_{ij}^{\cdot\cdot ab} = R_{ij}^{\cdot\cdot[ab]} \tag{42}$$

If we apply (40) and (36) to (34), the second line of (34) can be deduced at once.

Let us now turn to the identity (35). We raise the index l and form the contracted double dual according to the prescriptions (39) and (40). A straightforward but tedious calculation yields

$$\hat{\nabla}_k(e*R*^{k}_{\cdot i}) \equiv -e(2*R*^{l}_{\cdot j}S_{li}^{\cdot\cdot j} + P_j^{\cdot kl}R_{lik}^{\cdot\cdot\cdot j} + \tfrac{1}{2}\,G^{(jk)}Q_{ijk}) \tag{43}$$

an identity the two sides of which are separately projectively invariant. The alternative form of (43), which may look a bit nicer, is

$$\nabla_k(*R*^{k}_{\cdot i}) \equiv \Pi_{jk\cdot}^{\,\cdot\cdot l}R_{li}^{\cdot\cdot kj} - (\Pi_{[ij]k} + \Pi_{[j|k|i]} + \Pi_{k[ji]})\,R_m^{\cdot kmj} + \Pi_{ijk}R^{kj} \tag{44}$$

where

$$\Pi_{ij\cdot}^{\cdot\cdot k} := P_{ij\cdot}^{\cdot\cdot k} + \tfrac{1}{2}\,P_{[li]\cdot}^{\cdot\cdot l}\delta_j^k - \tfrac{1}{2}\,P_{[lj]\cdot}^{\cdot\cdot l}\delta_i^k \tag{45}$$

is a modification of the Palatini tensor.

The variation of the gravitational Lagrangian (3) with respect to the connection defines the Palatini tensor; variation with respect to the metric gives the symmetric part of the Einstein tensor (36). The importance of (34) for the divergence of the Palatini tensor is readily apparent. However, the divergence of the contracted double dual (43) of the curvature tensor does not directly relate to the divergence of $G^{(ij)}$. The resolution of this problem lies in the inherent limitation of the use of holonomic coordinates. The projectively invariant tensor $*R*^{ij}$ occurs naturally in the language of the anholomic (tetrad) formulation upon variation with respect to the tetrads.

§(4): *The Matter Lagrangian*

The symmetric energy-momentum tensor ${}^{\Gamma}\sigma^{ij}$ and the hypermomentum tensor $\Delta_{k}{}^{ji}$ are defined as variational derivatives of the matter Lagrangian with respect to the metric and the connection:

$$\delta \mathcal{L} \sim e(\tfrac{1}{2}\, \delta g_{ij}\, {}^{\Gamma}\sigma^{ij} - \delta \Gamma_{ij}{}^{k} \Delta_{k}{}^{ji}) \tag{46}$$

[cf. I (3.5) and I (3.6).] If only first-order derivatives of the matter fields ψ occur in $\mathcal{L}$, the canonical energy-momentum tensor and the symmetric tensor ${}^{\Gamma}\sigma^{ij}$ are related by

$$\mathcal{L}\delta_i^j - \frac{\partial \mathcal{L}}{\partial \partial_j \psi} \overset{\Gamma}{\nabla}_i \Psi = e\, {}^{\Gamma}\sigma_{i}{}^{j} + \hat{\nabla}_k (e \Delta_i{}^{jk}) \tag{47}$$

which is the prototype of the Belinfante-Rosenfeld symmetrization formula [23] for the energy-momentum tensor.

An alternative formulation of (46), fully equivalent to it, is obtained if the gravitational field is regarded as being represented by the metric, the *torsion*, and the *nonmetricity*. We then have (46) in the form

$$\delta \mathcal{L} \sim e(\tfrac{1}{2}\, \delta g_{ij} B^{ij} - \tfrac{1}{2}\, \delta Q_{ijk} \nu^{kji} + \delta S_{ij}{}^{k} \mu_{k}{}^{ji}) \tag{48}$$

where

$$eB^{ij} = e\, {}^{\Gamma}\sigma^{ij} + \hat{\nabla}_k (e \nu^{ijk}) \tag{49}$$

$$\nu_{ijk} = \Delta_{j(ik)} + \Delta_{i(kj)} - \Delta_{k(ji)} = \nu_{jik} \tag{50}$$

$$\mu_{ijk} = \Delta_{[ji]k} + \Delta_{[ik]j} - \Delta_{[kj]i} = -\mu_{ikj} \tag{51}$$

The tensor B^{ij} which couples to the metric in this formulation is a generalization of the Belinfante-Rosenfeld symmetrized energy-momentum tensor [23] (which couples to the metric in conventional GR). This is readily apparent when we consider that the Riemannian part of the connection (Christoffel symbols) is absent from the second and third terms of (48). Equation (49) can be cast in a form resembling the relation between the Belinfante tensor and the canonical tensor. Substituting the identity

$$\Delta^{ijk} \equiv \nu^{ijk} - \mu^{ijk} \tag{52}$$

in (49), we obtain by means of (47), on setting the nonmetricity to zero, the U_4 relation[6]

$$eB^{ij} = \left(\mathcal{L}g^{ij} - \frac{\partial \mathcal{L}}{\partial \partial_j \psi} \overset{\Gamma}{\nabla}_i \psi\right) + \hat{\nabla}_k (e\mu^{ijk}) \tag{53}$$

Other equivalent formulations of (46) are possible. For example, working with metric, *nonmetricity*, and *contortion*, we have

$$\delta \mathcal{L} \sim e[\tfrac{1}{2}\, \delta g_{ij}(B^{ij} - 2S_{kl}{}^{(i}{}_{.}\mu^{j)lk}) + \delta M_{kji} \Delta^{[ij]k} - \tfrac{1}{2}\, \delta Q_{kji} \Delta^{(ij)k}] \tag{54}$$

Thus the contortion and the nonmetricity couple, respectively, to the *spin current* and the *proper hypermomentum* current.

§(5): *The Riemannian Constraint*

The final term in the Lagrangian density (2) is the term containing the Riemannian constraint, which we can write in three alternative but equivalent ways:

$$\mathcal{C} := e \overset{\circ}{\Delta}{}_k{}^{ji} W_{ij.}{}^k \tag{55}$$

$$\mathcal{C} := e(\tfrac{1}{2}\, \overset{\circ}{\nu}{}^{kji} Q_{ijk} - \overset{\circ}{\mu}{}_k{}^{ji} S_{ij.}{}^k) \tag{56}$$

$$\mathcal{C} := e(-\overset{\circ}{\Delta}{}^{[ij]k} M_{kji} + \tfrac{1}{2}\, \overset{\circ}{\Delta}{}^{(ij)k} Q_{kji}) \tag{57}$$

The quantities with circle superscripts $^\circ$ are Lagrange multipliers. The relations between $\overset{\circ}{\Delta}$, $\overset{\circ}{\nu}$, and $\overset{\circ}{\mu}$ are of precisely the same form as those between the currents Δ, ν, and μ given in (50) and (51). The theory that has now been set up, obtained by independent variation of metric, connection, and Lagrange multipliers, is precisely GR, as was demonstrated in detail in I.

Because of the special role of the Weyl vector with regard to the projective invariance of $\mathcal{R}$ and the influence of this invariance on the process of relaxation of constraints, it will be found useful to separate out the Weyl-vector contribution in (56), and we then have

$$\mathcal{C} := e[\tfrac{1}{2}\, (\overset{\circ}{\nu}{}^{kji} - \tfrac{1}{4}\, g^{kj} \overset{\circ}{\nu}{}_l{}^{li})\, {}^{\dagger}Q_{ijk} - \overset{\circ}{\mu}{}_k{}^{ji}\, {}^{\dagger}S_{ij.}{}^k + \tfrac{1}{2}\, \overset{\circ}{\Delta}{}^i Q_i] \tag{58}$$

where $\overset{\circ}{\Delta}{}^i := \overset{\circ}{\Delta}{}_l{}^{li}$. In this form, the constraint is explicitly divided into a projectively invariant part involving ${}^{\dagger}Q_{ijk}$ and ${}^{\dagger}S_{ij.}{}^k$, and a projectively noninvariant part associated with the Weyl vector Q_i.

[6] As we recognize in (53), the classical Belinfante–Rosenfeld symmetrization of the energy-momentum tensor can be executed already in a U_4; see, however, Kopczyński [8] for a contrasting point of view.

§(6): *Nonzero Torsion: Einstein-Cartan-Sciama-Kibble Theory*

We first consider the consequences of relaxing the torsion constraint $S_{ij\cdot}{}^{k} = 0$ of GR. Thus we obtain a theory on a U_4 space-time, in which the connection is asymmetric, but the metric condition $\overset{\Gamma}{\nabla}_k g_{ij} = 0$ holds. This theory is identical with the EC theory [7]. The Lagrangian is given by (2), where the constraint term is in the form (56) with $\mathring{\mu}_{k\cdot}{}^{ij} = 0$. Variation of metric, connection, and $e\mathring{\nu}^{kji}$ leads to the field equations

$$Q_{ijk} = 0 \tag{59}$$

$$P^{ijk} = \kappa(\Delta^{ijk} - \mathring{\nu}^{ijk}) \tag{60}$$

$$eG^{(ij)} = \kappa\,[e\,{}^{\Gamma}\sigma^{ij} + \hat{\nabla}_k(e\mathring{\nu}^{ijk})] \tag{61}$$

Because of the symmetry of $\mathring{\nu}$, the second of these equations implies

$$P^{[ij]k} = \kappa\Delta^{[ij]k} \tag{62}$$

Thus the *spin* is dynamically coupled to the geometry. The *proper hypermomentum* $\Delta^{(ij)k}$ is not—it provides the constraint force that maintains the metric condition. Since the nonmetricity vanishes, (19) is just

$$P_{ij\cdot}{}^{k} = T_{ij\cdot}{}^{k} := S_{ij\cdot}{}^{k} + 2\delta_{[i}^{\ k} S_{j]l\cdot}{}^{l} \tag{63}$$

Therefore $P_{(ij)\cdot}{}^{k} = 0$ and (62) is simply

$$T^{ijk} = \kappa\Delta^{[ij]k} \tag{64}$$

The skew-symmetry of the Palatini tensor in a U_4 enables us to identify the constraint force in (60) as the proper hypermomentum

$$\mathring{\nu}^{ijk} = \Delta^{(ij)k} \tag{65}$$

Substituting this in (61) leads to

$$eG^{(ij)} = \kappa\,[e\,{}^{\Gamma}\sigma^{ij} + \hat{\nabla}_k(e\Delta^{(ij)k})] \tag{66}$$

Take the divergence of (62) and apply the identity (34) (note that, because the nonmetricity vanishes in a U_4, the carat derivative commutes with the raising and lowering of indices, and also $R_{ij(kl)} = 0$). We obtain

$$eR^{[ij]} = \kappa\hat{\nabla}_k(e\Delta^{[ij]k}) \tag{67}$$

Adding (66) and (67), and recalling the expression (47) for the canonical energy-momentum tensor, we have finally

$$G_i^{\cdot j} = \frac{\kappa}{e}\left(\mathcal{L}\delta_i^j - \frac{\partial\mathcal{L}}{\partial\partial_j\psi}\nabla_i\psi\right) \tag{68}$$

Equations (68) and (64) will be recognized as the field equations of the EC theory.

§(7): *Relaxation of the Metric Condition*

In this section, we consider the consequences of relaxing the metric condition in Einstein's theory, while preserving the symmetry of the connection. That is, we take $\mathring{\nu}^{kji} = 0$ in (56). The theory is clearly equivalent to the one obtained by independent variation of metric and connection in the Lagrangian density $(1/2\kappa)\,\mathcal{R} + \mathcal{L}$, where the connection is taken a priori to be symmetric. The field equations obtained from the variation of $e\mathring{\mu}$, connection, and metric are, respectively,

$$S_{ij.}{}^{k} = 0 \tag{69}$$

$$\frac{1}{\kappa} P_{k}{}^{\cdot ji} = \Delta_{k}{}^{\cdot ji} - \mathring{\mu}_{k}{}^{\cdot ji} \tag{70}$$

$$G^{ij} = \kappa\, {}^{\Gamma}\sigma^{ij} \tag{71}$$

Because of the skew-symmetry of $\mathring{\mu}$,

$$P_{k}{}^{\cdot (ji)} = \kappa \Delta_{k}{}^{\cdot (ji)} \tag{72}$$

and because of the vanishing torsion, equation (19) gives

$$P_{ij.}{}^{k} = \delta_{[i}{}^{k}\, {}^{\dagger}Q_{l]j.}{}^{l} - \delta_{[i}{}^{k} Q_{j]} \tag{73}$$

Therefore,

$$\tfrac{1}{2} Q_{i}{}^{\cdot jk} = - P_{i}{}^{(jk)} + g^{jk}(\tfrac{1}{2} P_{il}{}^{l} - \tfrac{1}{3} P^{l}{}_{.(il)}) + \tfrac{1}{3}(\delta^{j}_{i} P_{l}{}^{(kl)} + \delta^{k}_{i} P_{l}{}^{(jl)}) \tag{74}$$

Note that the right-hand side contains only the symmetric part, $P_{i}{}^{(jk)}$, of the Palatini tensor. Thus we obtain, from (74) and (72), an expression for the nonmetricity in terms of the part $\Delta_{k}{}^{\cdot (ji)}$ of the hypermomentum. The rest of the hypermomentum would not be dynamically active in producing the geometry. From the forms of (50), (51), and (52) we see that we could equally well say that, in such a theory, the current ν^{ijk} is dynamically active and the μ^{ijk} is not. On the other hand, they both contribute to the constraint force that keeps the connection symmetric:

$$\mathring{\mu}_{k}{}^{\cdot ji} = \Delta_{k}{}^{\cdot [ji]} + \tfrac{1}{3}(\delta^{j}_{k} \Delta_{l}{}^{\cdot (il)} - \delta^{i}_{k} \Delta_{l}{}^{\cdot (jl)}) \tag{75}$$

Unlike the EC theory, the theory presented here is not expected to have any physical significance, basically because the current ν, in contrast to the spin current, does not seem to play an active role under normal circumstances in the low-energy region, see Section 9. It was investigated purely as an illustrative example of the procedure of relaxing the constraints. The main point to be made is that it is a mathematically consistent theory. The reason for this is that the remaining constraint is *projectively noninvariant.*

In order to illustrate what happens when the remaining constraints, after relaxation, are projectively invariant, we consider an equally artificial theory in which we have an independent metric and *contortionless* connection. The remaining constraint in this case (vanishing contortion) is projectively invariant, and the characteristic peculiarity of $(1/2\kappa)\,\mathcal{R}$ as a Lagrangian density makes itself felt. The constraint term is now (57) with $\mathring{\Delta}^{(ij)k} = 0$, and $e\mathring{\Delta}^{[ij]k}$, the connection, and the metric are independently varied. We ignore the equation that comes from the metric, which is complicated and contributes nothing to the argument. The remaining field equations are

$$M_{ijk} = 0 \tag{76}$$

$$p^{kji} = \kappa(\Delta^{kji} - \mathring{\Delta}^{[kj]i}) \tag{77}$$

Hence

$$p^{(kj)i} = \kappa\Delta^{(kj)i} \tag{78}$$

The inconsistency is at once apparent. Since the Palatini tensor is traceless in its first two indices, such a theory would be possible only for matter with a vanishing dilation current $\Delta^i := \Delta_l{}^{li}$. Because the geometry is projectively invariant, it cannot respond to the degrees of freedom of the matter associated with projective transformations.

Since the contortion vanishes, equation (21b) shows that the Palatini tensor is symmetric, and equation (77) identifies the constraint force as the spin,

$$\mathring{\Delta}^{[kj]i} = \Delta^{[kj]i} \tag{79}$$

and only the proper hypermomentum $\Delta^{(kj)i}$ is dynamically active in determining the geometry. It gives rise to the Palatini tensor through (78) which determines the volume-preserving connection ${}^{\dagger}\Gamma$ through equation (24). *The Weyl vector remains undetermined*—a further consequence of the projective invariance of the remaining constraint. This therefore illustrates for the scalar curvature Lagrangian the principle: The remaining constraint on the geometry must not be projectively invariant.

§(8): *Single Vector Constraint*

The theory based on the Lagrangian density $(1/2\kappa)\,\mathcal{R} + \mathcal{L}$ without any constraint (fully independent metric and connection) in general does not determine the connection uniquely because of the projective invariance of $\mathcal{R}$; hence it is inconsistent from a physical point of view.[7] At least four degrees of freedom must be constrained, and these must *not* be projectively invariant.

Sandberg's [24] avoidance of this difficulty was an artificial prescription for making the matter Lagrangian projectively invariant. The proposal was to replace the covariant derivatives in $\mathcal{L}$ by covariant derivatives based on the projectively invariant connection

[7] The generalized Einstein–Cartan theory of [16] is of this type.

$$*\Gamma^k_{ij} = \Gamma^k_{ij} - \tfrac{2}{3} S_i \delta^k_j \tag{80}$$

Although this prescription leads to a *mathematically* consistent theory, there is no *physical* justification for the ad hoc hypothesis of a projectively invariant matter Lagrangian. In any case, the use of the connection (80) represents an arbitrary violation of the minimal coupling hypothesis without any direct geometrical significance.

Nevertheless, it is interesting to note, by way of an illustrative example, how Sandberg's theory fits into the scheme of metric-affine theory with constraints. It corresponds to the relaxation of all of the Riemannian constraint except the single vector constraint

$$\mathcal{C} = e\mathring{\mu}^i S_i \tag{81}$$

(which is not projectively invariant, so that the inconsistencies mentioned above are avoided). The resulting field equations are

$$S_i = 0 \tag{82}$$

$$P_k{}^{ji} = \kappa(\Delta_k{}^{ji} - \mathring{\mu}^{[i}\delta^{j]}_k) \tag{83}$$

$$G^{(ij)} = \kappa\ {}^{\Gamma}\sigma^{ij} \tag{84}$$

Contraction of (83) identifies the constraint force and the dilation current

$$\mathring{\mu}^i = \tfrac{2}{3}\,\Delta^i \tag{85}$$

Substituting back in (83) gives

$$P_k{}^{ji} = \kappa(\Delta_k{}^{ji} - \tfrac{2}{3}\,\delta^{[j}{}_k\Delta^{i]}) \tag{86}$$

Thus the Palatini tensor, and hence the contortion and the traceless part of nonmetricity, are determined by a traceless combination of the hypermomentum. Since $S_i = 0$, this determines the whole connection. The resulting theory is mathematically consistent. It is equivalent to Sandberg's theory; only the viewpoint is different. In the Sandberg formulation, the matter Lagrangian was artificially contrived to be projectively invariant, by coupling the matter to $*\Gamma$ instead of to Γ. In our formulation, true minimal coupling to Γ is maintained, but the constraint ensures that Γ is equal to $*\Gamma$. The implausibility of both versions of Sandberg's theory is highlighted by the fact that the dilation current is responsible for the constraint force that keeps the *torsion vector* (instead of the physically related Weyl vector) zero.[8]

[8] In the Weyl–Cartan space-time without contortion, (see I, Table I), the Weyl vector Q_i and the torsion vector S_i are proportional to each other, but this is a very special kind of space-time in which all other non-Riemannian parts of the connection are absent. In general, there is no relationship between the Weyl vector and the torsion vector in a metric-affine space-time.

In view of these criticisms, a more viable Sandberg-type prescription would be the adoption of the *volume-preserving connection* $^{\dagger}\Gamma$ for forming the covariant derivatives in the Lagrangian density $(1/2\kappa)\,\mathcal{R} + \mathcal{L}$, since this connection defined in (9) does have an important geometrical meaning which is related to the idea of projective invariance. The theory that then arises is identical to the one obtained by the relaxation of all constraints except

$$\mathcal{C} = \tfrac{1}{2}\, e\overset{\circ}{\Delta}{}^{i} Q_i \tag{87}$$

in (58). The field equations of this theory are

$$Q_i = 0 \tag{88}$$

$$P_{k}{}^{ji} = \kappa(\Delta_{k}{}^{ji} - \tfrac{1}{4}\,\delta^{j}_{k}\,\overset{\circ}{\Delta}{}^{i}) \tag{89}$$

$$eG^{(ij)} = \kappa\,[e^{\Gamma}\sigma^{ij} + \tfrac{1}{4}\;\partial_k(e\overset{\circ}{\Delta}{}^{k})\,g^{ij}] \tag{90}$$

Contraction of (89) shows that the constraint is provided by the dilation current:

$$\overset{\circ}{\Delta}{}^{i} = \Delta^{i} \tag{91}$$

Note that this constraint is part of the full Riemannian constraint found in I, equation (58), whereas the constraint (85) of the Sandberg theory is not. Because of (89) and (91), we have

$$P_{k}{}^{ji} = \kappa(\Delta_{k}{}^{ji} - \tfrac{1}{4}\,\delta^{j}_{k}\Delta^{i}) =: \kappa\overline{\Delta}_{k}{}^{ji} \tag{92}$$

Thus the whole of the volume-preserving connection $^{\dagger}\Gamma$ is determined by the traceless part of the hypermomentum. Substituting (91) in (90), the field equation coming from the metric takes the form

$$eG^{(ij)} = \kappa\,[e\,{}^{\Gamma}\sigma^{ij} + \tfrac{1}{4}\,g^{ij}\partial_k(e\Delta^{k})] \tag{93}$$

§(9): *The Full Metric-Affine Theory*

The theory with a single vector constraint ensuring the vanishing of the Weyl vector, in which the whole of the shear current becomes dynamically active in determining the geometry, see (92), and in which, according to (91), the dilation current provides the constraint force which keeps the Weyl vector to zero, is the nearest approach we can obtain to a theory with fully independent metric and affine properties, *as long as we insist on the hypothesis that the whole of the gravitational part of the Lagrangian is the curvature scalar.*

In the context of investigations in the *macroscopic* domain of physics, the Einstein theory appears to be completely adequate for explaining the experimental results, including the most recent experiments in lunar-laser-ranging [25], and the introduction of a connection with independent nonmetric aspects appears to be unwarranted. However, on the level of *microscopic* physics, the situation is quite different. Clearly, as we probe to smaller and smaller space-time separations, we naturally encounter the question of the interaction of geometry (the gravitational field) with the other fundamental forces of nature. We now know that a particle in the form of an extended object with orbital angular momentum, freely falling in a classical gravitational field, is subjected to Mathisson-type forces [26] that cause deviations from geodesic motion. Thus, in order to avoid the unlikely conclusion that the orbital angular momentum interacts with geometry but the intrinsic angular momentum does not, the constraint preventing the *spin* current from being dynamically active, at least, should be relaxed. Thus the generalization of GR to an EC-type theory in a U_4 space-time with nonvanishing torsion appears more or less inevitable.

The role of nonmetricity in physical systems has until recently been more obscure. However, the importance of scale invariance in electroweak interactions [27] suggests a relationship between the *dilation* current of matter and a nonvanishing Weyl vector. The tracefree part of the proper hypermomentum (*shear* plus *spin*) has recently been seen to provide the generators of the $SL(3, R)$ algebra possibly underlying the phenomenon of Regge trajectories (Hehl, Lord, and Ne'eman [5]).

The relaxation of the Riemannian constraint can be considered to take place in stages, as we go to observations involving higher energies, the relaxation of a constraint becoming significant at the characteristic energy at which the degree of freedom released by the relaxation becomes physically active. Thus, we would expect the spin current to become dynamically active at low energies, whereas the shear current would not become important until intermediate energies of the order of perhaps a few hundred MeV corresponding to the mass differences associated with the $\Delta J = 2$ excitations of the Regge trajectories. The scale invariance is associated with the zero mass limit, or equivalently, the ultra-relativistic limit corresponding to energies in excess of several tens of GeV. In view of these considerations, it becomes clear why an EC-type theory has physical significance, whereas the theory with nonmetricity but no torsion outlined in Section 6 does not—the torsion effects belong to a lower energy domain than the effects of nonmetricity. A theory with a single vector constraint, keeping $Q_i = 0$, like the one described at the end of the previous section, would be expected to describe the interaction between matter and geometry at intermediate energies.

The conclusion from the above arguments is that, at very high energies where scale invariance occurs, the dilation current becomes active and the interaction between matter and geometry would be described by a full metric-affine theory, without constraints. The metric and connection are then completely independent and all components of the hypermomentum current are dynamically active.

In order to construct such a theory, however, the curvature scalar $\mathcal{R}$ is not adequate as a gravitational Lagrangian. Equation (1) has to be replaced by a more general scalar density involving the metric and connection, and it should *not* be projectively invariant (or have any invariances other than coordinate transformation invariance). On the other hand, it should reduce to GR or to a theory experimentally indistinguishable from GR, when the complete Riemannian constraint is operative.

A simple solution was suggested by Hehl, Kerlick, and Von der Heyde [5]; we can add a projectively noninvariant Lagrangian for the Weyl vector. For example, by postulating that the Weyl vector is the field of a conventional massive boson: then the gravitational Lagrangian reads

$$\mathcal{O} = e\left(\frac{1}{2\kappa} g^{ij} R_{ij} + \alpha Q_{ij} Q^{ij} + \beta Q_i Q^i\right) \tag{94}$$

where $Q_{ij} := \partial_{[i} Q_{j]}$. Note that an electromagnetic-type Lagrangian $\beta = 0$ would still be invariant under the special projective transformations with $\lambda_i = \partial_i \lambda$, so that a single scalar constraint would remain (the divergence of the dilation current remains dynamically inactive).[9] The choice suggested by the above-mentioned authors was $\alpha = 0, \beta \neq 0$. A highly interesting alternative has been proposed recently by Papapetrou and Stachel [28]: they add the term, $g^{ij}\partial_i \Gamma_j$, to the curvature scalar, thereby also getting rid of the projective invariance.[10] Aldersley [29], in a most interesting investigation, has shown that, under certain suitable conditions, only W^2 terms are allowed to supplement $\mathcal{R}$ in (94). Other attractive options have been discussed by Ne'eman and Šijački [5].

Of course, an ad hoc prescription like (94) is not very convincing, and cannot be regarded as a final answer. A more fruitful approach might be to abandon $\mathcal{R}$ as a piece of the gravitational Lagrangian and to investigate other possibilities.

Recently an *anholonomic* geometrical framework for a gauge theory of the Poincaré group has been developed yielding a U_4 (with zero nonmetricity) as the appropriate space-time (see [30] and references given there). The class of Lagrangians leading to quasilinear second-order field equations for the gravitational potentials turns out to be a second-order polynomial in torsion and curvature. With a specific Lagrangian quadratic in torsion and curvature one finds, on imposing a teleparallelism constraint, a gravitational theory which is experimentally indistinguishable from GR [31]. The full theory in linear approxima-

[9] One simple modification of the curvature scalar $\mathcal{R}$, that does not suffer from the ad hoc postulate (94), is based upon the similarity between volume-preserving, and therefore metric-dependent, transformations of the connection and conformal transformations of the metric (Smalley [21]). The procedure leads to a metric-affine theory with an active dilation current. However, the Weyl vector is *constrained* to be equal to the gradient of the scalar conformal factor, and as a result is curl-free and thus not the most general possible Weyl vector.

[10] An alternative, and perhaps more natural method to arrive at the Papapetrou–Stachel Lagrangian, has been given recently by Stachel (private communication). He used the idea of transposition invariance (see Einstein [4], Appendix II) in constructing their Lagrangian.

tion (even without the metric constraint) embodies, in addition to the Newtonian potential, a rising confinement-type potential.

For the four-dimensional affine group this gauge theoretical anholonomic framework can be readily extended to the corresponding metric-affine spacetime (L_4, g) (Lord [5]) leading in this way to the full metric-affine theory in an anholonomic setup. The purely quadratic Lagrangian mentioned above is, if formulated in an (L_4, g), not projectively invariant and could be a possible substitute for our ansatz (94). This appears to us to be the most promising starting point for an attempt to unify gravity with high-energy physics.

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Note Added in Proof

For the variational principle one should also compare Bruzzo [32]. The choiçe of the Lagrangian in metric-affine gravity is discussed in the interesting articles by Kämpfer [33] and Sijacki [34].

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