THE METRIC-AFFINE GRAVITATIONAL THEORY AS THE GAUGE THEORY OF THE AFFINE GROUP

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The metric-affine gravitational theory is shown to be the gauge theory of the affine group, or equivalently, the gauge theory of the group GL(4,R) of tetrad deformations in a space–time with a locally Minkowskian metric. The identities of the metric-affine theory, and the relationship between them and those of general relativity and Sciama–Kibble theory, are derived.

There has been recently a great deal of interest in the application of the Yang–Mills idea to space–time symmetries. Kibble derived the Sciama–Kibble theory [1,2] by gauging the Poincaré subgroup of the infinitesimal coordinate transformations. The extension of this procedure to larger groups of infinitesimal coordinate transformations has been worked out by Harnad and Pettitt [3]. Poincaré gauge theories have also been given by von der Heyde [4] and Cho [5]. In von der Heyde's approach the translational subgroup is identified with the action of *parallel transport* of the physical fields. This leads to a much clearer understanding of the geometrical meaning of the Poincaré-gauge idea. The Poincaré-gauge idea has been extended to gauge theories of the de Sitter and conformal groups and the corresponding supersymmetries [6,7]. However, in view of the fact that the group SL(3,R) of deformations of hadronic matter can explain the observed structure of the Regge trajectories [8], the *affine group* would appear to be a physically more relevant extension of the Poincaré-group [9]. The affine gauge theory turns out to be identical with the metric-affine gravitational theory of Hehl et al. [10]. In the tetrad version of the metric-affine theory to be presented here, the tetrad (translational gauge potential) couples to the canonical energy–momentum density and the connection (gauge potential for tetrad deformations) couples to the canonical hypermomentum current density.

1. Gauging the tetrad deformations. The geometrical structure that we propose is the following. Consider a space–time with a metric and introduce a tetrad field (not necessarily orthonormal). A change of tetrad is specified by a position-dependent non-singular 4 × 4 matrix. Thus we have a local group GL(4,R) of tetrad "deformations", for which we postulate the existence of a gauge potential. We set up a lagrangian field theory and require it to be covariant for both coordinate transformations and tetrad deformations. Let $e_i{}^\alpha$ be the matrix of components of the tetrad and denote its inverse by $e_\alpha{}^i$ (holonomic or coordinate-based indices are denoted by $i, j, \ldots$ and anholonomic or tetrad-based indices are denoted by $\alpha, \beta, \ldots$). Let ϕ be the set of anholonomic components of a physical field, with the linear transformation law $\delta\phi = \epsilon_\beta{}^\alpha f_\alpha{}^\beta \phi$ under the infinitesimal tetrad deformation $\delta e_i{}^\alpha = \epsilon_\beta{}^\alpha e_i{}^\beta$ (fermion fields will correspond to an infinite-dimensional representation of GL(4,R), which has no finite-dimensional spinor representations [9]. The deformation gauge potentials $\lambda_{i\beta}{}^\alpha$ of GL(4,R) enter into the covariant derivative

$$\phi_{,i} := \partial_i \phi + \lambda_{i\beta}{}^\alpha f_\alpha{}^\beta \phi . \tag{1.1}$$

The deformation gauge fields are

$$R_{ij\alpha}{}^\beta := \partial_j \lambda_{i\alpha}{}^\beta - \partial_i \lambda_{j\alpha}{}^\beta + \lambda_{j\alpha}{}^\gamma \lambda_{i\gamma}{}^\beta - \lambda_{i\alpha}{}^\gamma \lambda_{j\gamma}{}^\beta . \tag{1.2}$$

The $\lambda_{i\beta}{}^{\alpha}$ are the anholonomic components of an affine connection, whose holonomic components are

$$\Gamma_{ik}{}^{j} := (\lambda_{i\beta}{}^{\alpha} e_k{}^{\beta} + \partial_i e_k{}^{\alpha}) e_{\alpha}{}^{j}. \tag{1.3}$$

Then $R_{ijk}{}^{l} := R_{ij\beta}{}^{\alpha} e_k{}^{\beta} e_{\alpha}{}^{l}$ are the components of the curvature tensor belonging to this connection. Under the simultaneous action of an infinitesimal coordinate transformation $x^i \to x'^i = x^i - \xi^i$ and an infinitesimal tetrad deformation $\epsilon_{\beta}{}^{\alpha}$, we have

$$\delta\phi = \xi^j \partial_j \phi + \epsilon_{\beta}{}^{\alpha} f_{\alpha}{}^{\beta} \phi,$$

$$\delta e_i{}^{\alpha} = e_j{}^{\alpha} \partial_i \xi^j + \xi^j \partial_j e_i{}^{\alpha} + \epsilon_i{}^{\alpha}, \tag{1.4}$$

$$\delta\lambda_{i\beta}{}^{\alpha} = \lambda_{j\beta}{}^{\alpha} \partial_i \xi^j + \xi^j \partial_j \lambda_{i\beta}{}^{\alpha} - \epsilon_{\beta}{}^{\alpha}{}_{,i}.$$

2. *Gravitational source identities.* Let $\mathcal{L}(g_{ij}, e_i{}^{\alpha}, \phi, \phi_{,i})$ be a lagrangian density for a set of matter fields ϕ. Since we do not require the tetrad to be orthonormal, the $e_i{}^{\alpha}$ and the metric components g_{ij} are *a priori* independent. We have 10 + 16 + 64 = 90 "gravitational" components g_{ij}, $e_i{}^{\alpha}$, $\lambda_{i\beta}{}^{\alpha}$. The total lagrangian density will be of the general form $\mathcal{V} + \mathcal{L}$, where $\mathcal{V}$ is the gravitational lagrangian density, constructed from the 90 gravitational components and their derivatives. We study the structure that arises from the requirement that the theory shall be covariant for coordinate transformations and tetrad deformations. Note that our results are *not* dependent on the actual functional form of $\mathcal{V}$.

We define *sources* of the gravitational components:

$$\mathcal{G}_{ij} := 2\delta\mathcal{L}/\delta g^{ij}, \quad \mathcal{G}_i{}^{\alpha} := \delta\mathcal{L}/\delta e_{\alpha}{}^{i}, \quad \mathcal{G}_{\alpha}{}^{\beta i} := \delta\mathcal{L}/\delta\lambda_{i\beta}{}^{\alpha}, \tag{2.1}$$

and we also define

$$\mathcal{F} := \delta\mathcal{L}/\delta\phi. \tag{2.2}$$

The transformation laws (1.4) are supplemented by the transformation law of the metric:

$$\delta g_{ij} = \xi_{i,j} + \xi_{j,i}. \tag{2.3}$$

(Here the definition of the comma derivative has been extended – when operating on holonomic sets of components, the Christoffel symbol derivative is implied.) The required covariance properties lead to the following 4 + 16 identities satisfied by the source – an *energy–momentum identity* coming from coordinate invariance and a *hypermomentum identity* coming from invariance under tetrad deformations:

$$(\mathcal{G}_i{}^{j} + \mathcal{G}_i{}^{j})_{,j} - \mathcal{G}_{\alpha}{}^{j} e_j{}^{\alpha}{}_{,i} + \mathcal{G}_{\alpha}{}^{\beta j} R_{ji\beta}{}^{\alpha} + \mathcal{F} \cdot \phi_{,i} = 0, \tag{2.4}$$

$$\mathcal{G}_{\alpha}{}^{\beta i}{}_{,i} - \mathcal{G}_{\alpha}{}^{\beta} + \mathcal{F} \cdot f_{\alpha}{}^{\beta} \phi = 0. \tag{2.4}$$

Of course, a similar treatment of the *total* lagrangian density $\mathcal{V} + \mathcal{L}$ will lead to analogous identities satisfied by the field equations. We therefore have 90 gravitational equations satisfying 20 identities, so the gravitational field of this gauge theory has 70 dynamically independent components. The twenty degrees of freedom associated with the reference system can be eliminated by imposing four coordinate conditions and sixteen restrictions on the tetrad. This corresponds to the fact that the tetrad can be chosen arbitrarily.

If $\mathcal{L}$ is a scalar density for coordinate transformations and invariant for tetrad deformations, then $\mathcal{G}_i{}^{j} + \mathcal{G}_i{}^{j}$ is the *canonical* energy–momentum density and $\mathcal{G}_{\alpha}{}^{\beta j}$ is the *canonical* hypermomentum current density:

$$\begin{aligned} \mathcal{G}_i{}^{j} + \mathcal{G}_i{}^{j} &= (\partial\mathcal{L}/\partial\partial_j\phi)\phi_{,i} - \delta_i{}^{j}\mathcal{L} =: \sqrt{-g}\,\Sigma_i{}^{j}, \\ \mathcal{G}_{\alpha}{}^{\beta j} &= (\partial\mathcal{L}/\partial\partial_j\phi) f_{\alpha}{}^{\beta}\phi. \end{aligned} \tag{2.6}$$

3. *Special gauge conditions.* Let us choose the tetrad so that $e_i{}^{\alpha} = \delta_i{}^{\alpha}$. We refer to this as the *holonomic gauge condition.* It is preserved by the simultaneous action of a coordinate transformation ξ^i and a tetrad deformation $\epsilon_{\beta}{}^{\alpha} = -\partial_{\beta}\xi^{\alpha}$. All indices (latin or greek) can now be regarded as holonomic, and the gravitational field is described by a metric g_{ij} and a connection $\Gamma_{ij}{}^{k}$:

$$\mathcal{G}_{ij} = 2\delta\mathcal{L}/\delta g^{ij}, \quad \mathcal{G}_k{}^{ji} = \delta\mathcal{L}/\delta\Gamma_{ij}{}^{k}. \tag{3.1}$$

We have 10 + 64 gravitational equations satisfying 4 identities (obtained by eq. (2.5) as a *definition* of $\mathcal{G}_{\alpha}{}^{\beta}$ and substituting this into eq. (2.4)). We now have the L4(*g*) framework of the metric-affine theory.

Another kind of special gauge is an *orthonormal gauge*, obtained by requiring the tetrad to be orthonormal:

$$g^{ij} = e_\alpha{}^i e_\beta{}^j \eta^{\alpha\beta}\,, \tag{3.2}$$

where $\eta^{\alpha\beta}$ is the matrix of the Minkowski metric. When working within this framework, the group GL(4,R) of tetrad deformations is restricted to its Lorentz subgroup. The metric components are now *not* independent of the tetrad, and we have 16 + 64 gravitational equations satisfying 4 + 6 identities. In this formulation, the tetrad is not coupled to $\mathcal{G}_i{}^\alpha$ as in eq. (2.1), but to the canonical energy–momentum density $\sqrt{-g}\,\Sigma_i{}^\alpha$. The source identities are

$$\begin{aligned}(\sqrt{-g}\Sigma_i{}^j)_{,j} - (\mathcal{G}_\alpha{}^{\beta i}{}_{,i} + \mathcal{F}\cdot f_\alpha{}^\beta\phi) e_\beta{}^j e_j{}^\alpha{}_{,i}&\\ + \mathcal{G}_\alpha{}^{\beta j} R_{ji\beta}{}^\alpha + \mathcal{F}\cdot\phi_{,i} = 0\,,&\end{aligned} \tag{3.3}$$

$$\eta^{\gamma[\alpha}\mathcal{G}_\gamma{}^{\beta]i}{}_{,i} - \sqrt{-g}\Sigma^{[\alpha\beta]} + \mathcal{F}\cdot f^{[\alpha\beta]}\phi = 0\,. \tag{3.4}$$

It is to be emphasised that the use of different special gauges gives different but *equivalent* formulations of the same theory, namely the metric-affine gravitational theory.

4. Constraints on the connection. In the previous section, we have assumed that all the components are dynamically independent fields. However, if instead we choose to construct some or all of these quantities from the metric, the tetrad, and their derivatives, we shall obtain an alternative theory, more restricted than the general metric-affine theory. For example all the $\lambda_{i\beta}{}^\alpha$ could be constructed according to

$$\lambda_{i\beta}{}^\alpha = (\partial_i e_\beta{}^j + \{{}^{\,j}_{ik}\} e_\beta{}^k) e_j{}^\alpha\,,$$

instead of introducing them as independent fields. This is equivalent to introducing the *constraint*

$$\Gamma_{ik}{}^j - \{{}^{\,j}_{ik}\} \equiv e_\alpha{}^j e_k{}^\alpha{}_{,i} = 0$$

on the metric-affine theory [11]. Working in the holonomic gauge, we have just 10 gravitational equations satisfying 4 identities. We have clearly obtained the riemannian framework of Einstein's gravitational theory. The metric is now not coupled to $\mathcal{G}^{ij}$, but to

$$\begin{aligned}\mathcal{B}^{ij} := \sqrt{-g}\Sigma^{ij} + (\mathcal{G}^{[il]j} + \mathcal{G}^{[jl]i} + \mathcal{G}^{[ji]l})_{,l}&\\ - \mathcal{F}\cdot f^{ij}\phi\,,&\end{aligned} \tag{4.1}$$

and the source identities reduce to

$$\mathcal{B}_{i,j}^{\,j} + \mathcal{F}\cdot\phi_{,i} + (\mathcal{F}\cdot f_i{}^j\phi)_{,j} = 0\,. \tag{4.2}$$

When the field equations $\mathcal{F} = 0$ are satisfied, we have the usual energy–momentum conservation for the symmetrised Belinfante energy–momentum. The skewsymmetrised hypermomentum current density appearing in eq. (4.1) is of course the spin current density.

Consider now the less severe constraint $\lambda_{i(\alpha\beta)} = \frac{1}{2}\partial_i(g_{jk}e_\alpha{}^j e_\beta{}^k)$ which is equivalent to the vanishing of *non-metricity*:

$$Q_{ijk} := \partial_k g_{ij} - 2\Gamma_{k(ij)} = 0\,. \tag{4.3}$$

Only the 24 components $\lambda_{i[\alpha\beta]}$ of the gauge potential are now dynamically independent. In an orthonormal gauge, these are the components of the potential for the Lorentz rotations of the tetrad; we have the U_4 formalism of Poincaré gauge theory (or Sciama–Kibble theory) with 16 + 24 gravitational equations satisfying 4 + 6 identities. The source identities are obtained by imposing the constraint (4.3) on eqs. (3.3) and (3.4). We find

$$\begin{aligned}(\sqrt{-g}\Sigma_\alpha{}^j)_{,j} + 2\sqrt{-g}\Sigma_\beta{}^j S_{j\alpha}{}^\beta + \mathcal{G}_{[\beta\gamma]}{}^j R_{j\alpha}{}^{\gamma\beta}&\\ + \mathcal{F}\cdot\phi_{,i} = 0\,,&\end{aligned} \tag{4.4}$$

$$\mathcal{G}^{[\alpha\beta]j}{}_{,j} - \sqrt{-g}\Sigma^{[\alpha\beta]} + \mathcal{F}\cdot f^{[\alpha\beta]}\phi = 0\,, \tag{4.5}$$

where $S_{ij}{}^k := \Gamma_{[ij]}{}^k$ is the torsion. With $\mathcal{F} = 0$, these are the identities given by von der Heyde [4] and Kibble [1].

5. The affine gauge theory. The transformations (1.4) were introduced as combined coordinate transformations and GL(4,R) gauge transformations. They can be reinterpreted as affine gauge transformations. That is, as transformations of a gauged *inhomogeneous* linear group. The translations generate parallel transport of the physical fields and the tetrad is the gauge potential for the translation subgroup. To establish this, consider first the formalism for an *internal* symmetry group G, with generators f_A, acting as a gauge group. Let ϕ be a set of physical fields transforming linearly under the group, $\delta\phi = \epsilon\phi := \epsilon^A f_A\phi$, with covariant derivative $\phi_{;i} := \partial_i\phi + \lambda_i\phi$, where $\lambda_i := \lambda_i{}^A f_A$

is the gauge potential for G. The gauge fields will be $F_{ij} := \partial_j \lambda_i - \partial_i \lambda_j + [\lambda_j \lambda_i] = F_{ij}{}^A f_A$. Under the simultaneous action of an infinitesimal coordinate transformation $x^i \to x'^i = x^i - \xi^i$ and an infinitesimal gauge transformation with parameters ϵ^A, we have

$$\delta\phi = \xi^j \partial_j \phi + \epsilon\phi, \quad \delta\lambda_i = \lambda_j \partial_i \xi^j + \xi^j \partial_j \lambda_i - \epsilon_{;i}, \tag{5.1}$$

or, in terms of the covariant derivative,

$$\delta\phi = \xi^i \phi_{;i} + \zeta\phi, \quad \delta\lambda_i = \xi^j F_{ij} - \zeta_{;i}, \tag{5.2}$$

where $\zeta := \epsilon - \xi^i \lambda_i$. The transformation associated with the parameters ξ^i is now interpreted *actively* as a parallel transport of the fields ϕ, so that eq. (5.2) represents a parallel transport together with a gauge transformation with parameters ζ^A. Now let $\mathcal{L}(g_{ij}, \lambda_i^A, \phi, \partial_i\phi)$ be a lagrangian density for the matter fields ϕ. Covariance of its field equations under eq. (5.2) implies the source identities

$$\mathcal{G}_i{}^j{}_{;j} + \mathcal{G}_A{}^j F_{ji}{}^A + \mathcal{F}\cdot\phi_{;i} = 0, \quad \mathcal{G}_A{}^i{}_{;i} + \mathcal{F}\cdot f_A \phi = 0, \tag{5.3}$$

where $\mathcal{G}_A{}^i := \delta\mathcal{L}/\delta\lambda_i^A$.

Now let G be the 20-parameter affine group defined by

$$[f_\alpha{}^\beta, f_\gamma{}^\delta] = \delta_\gamma{}^\beta f_\alpha{}^\delta - \delta_\alpha{}^\delta f_\gamma{}^\beta,$$
$$[f_\alpha{}^\beta, f_\gamma] = \delta_\gamma{}^\beta f_\alpha, \quad [f_\alpha, f_\gamma] = 0, \tag{5.4}$$

but identify the "internal translations" with *parallel transport*. That is, $\zeta^\alpha = -\xi^j \lambda_j^\alpha$ or $\epsilon^\alpha = 0$. This restricts the identities (5.3) to the set

$$\mathcal{G}_i{}^j{}_{;j} + \mathcal{G}_A{}^j F_{ji}{}^A + \mathcal{F}\cdot\phi_{;i} - \lambda_i^\alpha(\mathcal{G}_\alpha{}^j{}_{;j} + \mathcal{F}\cdot f_\alpha \phi) = 0,$$
$$\mathcal{G}_\alpha{}^{\beta j}{}_{;j} + \mathcal{F}\cdot f_\alpha{}^\beta \phi = 0. \tag{5.5}$$

If we now write $e_i{}^\alpha := -\lambda_i{}^\alpha$ and introduce a *comma derivative* for the covariant differentiation associated with the homogeneous part of G, we find that eq. (5.1) (with $\epsilon^\alpha = 0$) is identical with eq. (1.4) and eq. (5.5) is identical with the set (2.4), (2.5). The gauge fields for the homogeneous part of the group and for the translations are respectively the curvature and the torsion:

$$F_{ij\beta}{}^\alpha = R_{ij\beta}{}^\alpha, \quad F_{ij}{}^\alpha = 2S_{ij}{}^\alpha. \tag{5.6}$$

Thus we have established that the metric-affine theory is an affine gauge theory, in the same sense that the Kibble–Sciama theory is a Poincaré gauge theory (the only difference is that the metric appears as an extraneous field, whereas in the Poincaré gauge theory it is determined by the translational gauge potentials).

As we have seen, the affine extension of Poincaré gauge theory is particularly straightforward – it is devoid of the additional complications (second-order frame structure) of the conformal gauge theories. Moreover, the affine group contains both scale transformations and the group SL(3,R) whose Lie algebra (generated by hypermomentum) appears to be responsible for the basic structure of the Regge trajectories. This is a strong indication that the metric-affine gravitational theory (which goes over into Einstein's theory in the "macroscopic" limit) is the appropriate extension of Einstein's theory in the "microscopic" domain.

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