HYPERMOMENTUM AND THE MICROSCOPIC VIOLATION OF THE RIEMANNIAN CONSTRAINT IN GENERAL RELATIVITY

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We derive Einstein's field equation by means of a metric-affine variational principle with an explicit Riemannian constraint. The corresponding Lagrange multiplier, the hypermomentum current, should be a measure for the microscopic violation of the constraint. We relax the Riemannian constraint and arrive at the metric-affine theory of gravitation.

1. *Metric-affine theory of gravitation.* In the recently proposed new metric-affine theory of gravitation [1–4], the gravitational field is described by the metric g_{ij} ($= g_{ji}$) and the linear connection Γ_{ij}^{k} ($\neq \Gamma_{ji}^{k}$) of spacetime. Both the momentum current ${}^{\Gamma}\sigma^{ij}$, conventionally called the energy-momentum tensor, and the newly recognized hypermomentum current $\Delta_k{}^{ji}$ act as sources of the gravitational field. If $\mathcal{V}$ $(g, \partial g, \Gamma, \partial\Gamma)$ is the gravitational field Lagrangian, then the two field equations read [‡1] (k = gravitational constant)

$$\frac{\delta \mathcal{V}}{\delta g_{ij}} = -\, ke\; {}^{\Gamma}\sigma^{ij}, \quad \frac{\delta \mathcal{V}}{\delta \Gamma_{ij}^{k}} = 2ke\Delta_k{}^{ji}\,. \tag{1, 2}$$

The first field equation with its 10 independent components is the analog of Einstein's equation, the second field equation has 64 independent components and generalizes the Christoffel relation $\Gamma_{ij}^{k} - \{{}_{ij}^{k}\} = 0$ known from conventional general relativity theory (GR).

We have shown that the hypermomentum current may be split into the intrinsic *dilation* current $\Delta^{i} := \Delta_l{}^{\cdot li}$, the intrinsic *shear* current $\bar{\Delta}^{ijk} := \Delta^{(ij)k} - g^{ij}\Delta^{k}/4$, and the *spin* current $\tau^{ijk} := \Delta^{[ij]k}$. Furthermore, GR is contained in eqs. (1, 2) for vanishing hypermomentum, provided we take [1, 2]

$$\frac{\mathcal{V}}{e} = g^{ij}R_{kij}{}^{k} + \beta Q_i Q^{i} \tag{3}$$

as gravitational field Lagrangian. β is a dimensionless universal constant ($\beta \neq 0$), $R_{ijk}{}^{l}$ the curvature tensor and Q_i the Weyl vector (see below). If only the shear and the dilation currents vanish, then we recover the U_4 theory of gravitation (see [6]), which is now an established theory. Since the metric-affine theory encompasses the Einstein and the U_4 limit, it is a well-defined and viable extension of GR. Additionally, its structure, in particular the coupling of hypermomentum to the linear connection, looks very convincing.

In the following we would like to present more evidence in favor of the metric-affine theory.

2. *Metric-affine variational principle with Riemannian constraint.* We analyze conventional GR with the help of a variational principle with independent metric and connection. We force the geometry of spacetime to stay Riemannian, however. For more details and references compare [7], see in this context also the imporant work of Kopczyński [8] and Trautman [9].

The model of spacetime is an (L_4, g) with independent metric and connection. Define the *torsion* tensor $S_{ij}{}^{k} := \Gamma_{[ij]}^{k}$, the *nonmetricity* tensor $Q_{ijk} := \overset{\Gamma}{\nabla}_i g_{jk}$, the *Weyl vector* $Q_i := Q_{il}{}^{l}/4$ and the traceless nonmetricity $\bar{Q}_{ijk} := Q_{ijk} - Q_i g_{jk}$. Then the linear connection may be written as

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[‡1] Our conventions are. Latin indices i, j . run from 0 to 3, Greek indices α, β . from 1 to 3. Symmetrization and antisymmetrization are denoted by ( ) and [ ], respectively. The definition sign is $:=$. Covariant differentiation with respect to the asymmetric connection Γ_{ij}^{k} is denoted by $\overset{\Gamma}{\nabla}_k$, partial differentiation by ∂_k. Furthermore, we use a tensor $\Delta_{jil}^{abc} = \delta_j^a\delta_i^b\delta_l^c + \delta_i^a\delta_l^b\delta_j^c - \delta_l^a\delta_j^b\delta_i^c$, producing permutations of the Christoffel type. Compare always Schouten [5]. $e = |\det. g_{ij}|^{1/2}$.

$$\Gamma^k_{ij} = g^{kl}\Delta^{abc}_{jil}(\partial_a g_{bc}/2 - S_{abc} + Q_a g_{bc}/2 + \bar{Q}_{abc}/2)\,. \tag{4}$$

The first term represents the Christoffel symbol $\{^k_{ij}\}$ of GR.

Let $\mathcal{L} = eL\,(\Psi, \overset{\Gamma}{\nabla}\Psi, g)$ be the material Lagrangian minimally coupled to the (L_4, g) and $R_{ijk}{}^l(\Gamma)$ the curvature tensor of the (L_4, g). Consider the variational principle

$$\delta_{g,\Gamma,\mathring{\Delta},\Psi}\int d^4x\, e\,[g^{ij}R_{kij}{}^k + C_1 + \mathrm{L}(\Psi, \overset{\Gamma}{\nabla}\Psi, g)] = 0 \tag{5}$$

with the constraint term

$$C_1 \;\; = \mathring{\Delta}_k{}^{ji}(\Gamma^k_{ij} - \{^k_{ij}\}) \tag{6}$$

$\mathring{\Delta}_k{}^{ji}$ is a Lagrange multiplier. We define $\sigma^{ij} := (2/e)\delta\mathcal{L}/\delta g_{ij}$ as the momentum current and $\Delta_k{}^{ji} := (-1/e)\delta\mathcal{L}/\delta\Gamma_{ij}{}^k$ as the hypermomentum current. In eq. (5) variation with respect to $\mathring{\Delta}$ and Γ gives

$$\Gamma^k_{ij} = \{^k_{ij}\}, \qquad \mathring{\Delta}_k{}^{ji} = \Delta_k{}^{ji}\,. \tag{7, 8}$$

In determining the Lagrange multiplier in eq. (8), we have already used eq. (7). Variation in eq. (5) with respect to g and using eqs. (7, 8) leads to the Einstein equation (G_{ij} = Einstein tensor)

$$\frac{1}{k}G^{ij}(\{\,\}) = \sigma^{ij} + \overset{\{\}}{\nabla}_k(\Delta^{j(ki)} + \Delta^{i(jk)} - \Delta^{k(ij)})\,. \tag{9}$$

The right-hand side of eq. (9) is the conventional metric energy-momentum tensor of GR, see [7].

Instead of the constraint term C_1 in eq. (5), we could have used the alternative expression [‡2]

$$C_2 := 2\mathring{\nu}^i Q_i + \tfrac{1}{2}\mathring{\nu}^{kji\dagger}Q_{ijk} + \mathring{\mu}_k{}^{ji\dagger}S_{ij}{}^k \tag{10}$$

with the multipliers $\mathring{\nu}^i$, $\mathring{\nu}^{kji}$, and $\mathring{\mu}^{kji}$. Observe that $\mathring{\nu}_k{}^{ki} \equiv 0$, $\mathring{\nu}^{k[ji]} \equiv 0$, $\mathring{\mu}_k{}^{(ji)} \equiv 0$. The nonmetricity and the torsion in eq. (10) refer to the new volume preserving connection

$$^\dagger\Gamma^k_{ij} = \Gamma^k_{ij} - \tfrac{1}{2}Q_i\delta^k_j\,. \tag{11}$$

Note $\bar{Q} = {}^\dagger Q$. The conditions ($Q_i = 0$, ${}^\dagger Q = 0$, ${}^\dagger S = 0$) are necessary and sufficient for $\Gamma = \{\,\}$. Consequently C_2 is equivalent to C_1. In particular we find

$$\Delta^{ijk} = g^{ij}\mathring{\nu}^k/4 + \mathring{\nu}^{ijk} - \mathring{\mu}^{ijk}\,. \tag{12}$$

In both ways, with C_1 or with C_2, respectively, we recover the field equation of Einsteinian GR. We collect or results:

[GR] plus [metric-affine way of looking at Riemannian geometry]
→ [constraint "force", the Lagrangian of which is C_1 or C_2] . (13)

3. *Meaning of the Lagrange multiplier.* In sec. 2 we only rewrote GR in a different mathematical framework. Nothing happened from a physical point of view.

But it is known from classical mechanics that a Lagrange multiplier is closely related to the constraint force which upholds the constraint. That is, C_1 or C_2, respectively, are the Lagrangians representing these constraint forces in relation (13). It is consistent with this interpretation that on the right-hand side of the Einstein equation (9) the Lagrange multiplier supplies energy to the source of the gravitational field via the hypermomentum Δ^{ijk} (compare eqs. 8, 12). Consequently we may state that

$$\Delta_k{}^{ji} \text{ keeps } \Gamma^k_{ij} \text{ Riemannian.} \tag{14}$$

Broadly speaking, for matter with non-vanishing hypermomentum, there exists the problem of confining this matter within the Riemannian spacetime [‡3]. Thereby we arrive at a natural interpretation of the Lagrange multiplier $\mathring{\Delta}_k{}^{ji}$ and the hypermomentum current $\Delta_k{}^{ji}$.

But we can go one step further: no constraint in physics is completely rigid; the notion of rigidity appears only within the domain of some approximation. Moreover, rigid spacetime structures are in any case contrary to the spirit of GR (Einstein [10], pp. 36, 94). If one looks into the formalism of classical mechanics, one finds that the Lagrange multiplier is "*. . . a measure of the microscopic violation of the equation*

[‡2] The constraints ${}^\dagger S_{ij}{}^k = 0$ and ${}^\dagger Q_{ijk} = 0$ are both projectively invariant, in contrast to the constraint $Q_i = 0$, which breaks projective invariance.

[‡3] This statement needs some qualification, see [7].

of constraint" (Lanczos [11], p. 144). Thus, we expect that hypermomentum, which is a Lagrange multiplier of order $\hbar$, is a measure of the microscopic violation of the Riemannian constraint in GR.

We expect a violation of the metric constraint $Q = 0$ and of the symmetry constraint $S = 0$ at the same time, since the arguments advanced above do not distinguish between these two constraints. In [1] we argued that allowing non-metric spacetimes ($Q \neq 0$) in the way we do it, should not lead to difficulties. Hayashi [12] reconsidered our arguments but doubted our conclusions. We hope to have shown that there seems to be no way around a microscopic violation of the $Q = 0$ constraint, provided we have a non-vanishing intrinsic dilation or shear current.

4. *Relaxation of the Riemannian constraint.* In analogy to classical mechanics, we propose to relax the Riemannian constraint. In going over from rigid body dynamics to continuum physics, the constraint force keeping the body rigid becomes a real intrinsic physical force (or rather stress). In the same way, during the relaxation process, hypermomentum loses its passive role that it had within GR and becomes a new *source* of the gravitational field in the metric-affine theory of gravitation.

If we take the variational principle (5) with the constraint term C_2 from eq. (10), then we can relax ${}^{\dagger}Q = 0$ and ${}^{\dagger}S = 0$ straightforward by just dropping the corresponding terms in the Lagrangian. However we note that $g^{ij}\,{}^{\dagger}R_{kij}{}^{k} = g^{ij}R_{kij}{}^{k}$. Hence the relaxation of the remaining constraint, $Q_i = 0$, requires a new piece in the gravitational field Lagrangian depending on Q_i, otherwise we run into inconsistencies. One possible choice which is near at hand, is the choice of a Q^2-term, as we did in eq. (3). But this point needs further investigations.

Consequently in relaxing ${}^{\dagger}Q = 0$ and ${}^{\dagger}S = 0$ we can just take the analog of the usual Hilbert-Einstein field Lagrangian; however, in relaxing $Q_i = 0$, we need a new physical principle. This is suggestive since relaxation of $Q_i = 0$ may be related to the mass-zero limit of matter.

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