A theorem on stress–energy tensors

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(Received 6 March 1975)

The equality of the symmetrized Noether stress–energy tensor (Belinfante's tensor) and the canonical stress–energy tensor (functional derivative of the Lagrangian density with respect to the metric) is established by methods based on the formalism of tetrads and Ricci rotation coefficients. The result holds for any Lagrangian which contains no derivatives of the fields higher than first order.

The equality of Belinfante's symmetrized stress–energy (SE) tensor and the canonical SE tensor (functional derivative of the Lagrangian density with respect to the metric) was demonstrated for integral spin fields by Rosenfeld.[1] For fields with half-integral spin it is not immediately clear how the canonical SE tensor should be defined. Goedecke[2] has shown that, subject to a certain prescription for carrying out the variation of the metric, Rosenfeld's equality holds for the Dirac field and for the coupled Maxwell and Dirac fields. Goedecke conjectured that a general proof of the equality for any field should be possible. The proof presented here is based on the tetrad formalism,[3-5] in which the Noether SE tensor and the Noether spin tensor are defined as functional derivatives of the Lagrangian density with respect to the tetrad components and the Ricci rotation coefficients.

Let $L(\phi, \partial_\mu\phi)$ be the Lagrangian of a set of fields ϕ in a Cartesian coordinate system in Minkowski spacetime. We generalize it to a Lagrangian density $\mathfrak{L}$ in a curvilinear coordinate system in the following way.[3] Introduce a tetrad h^α_μ and convert all the coordinate based indices $(\mu, v, \cdots)$ on ϕ to tetrad based indices $(\alpha, \beta, \cdots)$, by contractions with h^α_μ or its inverse h^μ_α. Introduce a set of Ricci rotation coefficients $\lambda^{\alpha\beta}{}_\mu$ for the purpose of constructing a derivative of ϕ that is covariant for coordinate transformations and spacetime dependent Lorentz rotations of the tetrad. Then

$$\mathfrak{L}(\phi, \partial_\mu\phi, h^\alpha_\mu, \lambda^{\alpha\beta}{}_\mu) = hL(\phi, \phi_\alpha) \tag{1}$$

is the required generalization, invariant under tetrad rotations and a scalar density of weight 1 under coordinate transformations. We have used the notation

$$h = |h^\alpha_\mu| = (-g)^{1/2} \tag{2}$$

and

$$\phi_\alpha = h^\mu_\alpha(\partial_\mu\phi + \tfrac{1}{2}\lambda^{\alpha\beta}{}_\mu G_{\alpha\beta}\phi). \tag{3}$$

The quantity in brackets in (3) is the *covariant derivative* ϕ_μ and the $G_{\alpha\beta}$ are the constant matrices which generate the Lorentz rotations in the ϕ-representation.

Two tensor densities t^μ_α and $s^\mu{}_{\alpha\beta}$ are defined as the functional derivatives of $\mathfrak{L}$ with respect to h^α_μ and $\lambda^{\alpha\beta}{}_\mu$. That is, for the infinitesimal variations

$$\delta h^\alpha_\mu = \zeta^\alpha_\mu, \quad \delta\lambda^{\alpha\beta}{}_\mu = \zeta^{\alpha\beta}{}_\mu \tag{4}$$

we have

$$\delta\mathfrak{L} \sim t^\mu_\alpha \zeta^\alpha_\mu + \tfrac{1}{2}s^\mu{}_{\alpha\beta}\zeta^{\alpha\beta}{}_\mu$$

(where ~ denotes that all *divergences* have been omitted from an equation).

Suppose the original tetrad was $h^\alpha_\mu = \delta^\alpha_\mu$ before the variation (and hence $g_{\mu\nu} = \eta_{\mu\nu}$, i.e., the coordinate system was Cartesian), and suppose also the Ricci rotation coefficients were zero. Then, when we work only to first order in the infinitesimal quantities, the distinction between tetrad based indices and coordinate based indices is not relevant. All vector indices are raised and lowered with the Minkowski metric.

Substitute for the variations in (5) those brought about by an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$:

$$\zeta^\alpha_\mu = -\partial_\mu\xi^\alpha, \quad \zeta^{\alpha\beta}{}_\mu = 0 \tag{6}$$

(to first order). Integrate (5) over a region of four-space on the boundary of which ξ^α vanishes and apply Gauss's theorem. We obtain the identity

$$\partial_\mu t^\mu_\alpha = 0 \tag{7}$$

in the Cartesian system. Similarly, substitute the variations brought about by an infinitesimal tetrad rotation with parameters $\lambda_{\alpha\beta} = -\lambda_{\beta\alpha}$:

$$\zeta_{\alpha\beta} = \lambda_{\alpha\beta}, \quad \zeta^{\alpha\beta}{}_\mu = \partial_\mu\zeta^{\alpha\beta}, \tag{8}$$

and we obtain the identity

$$\partial_\mu s^\mu{}_{\alpha\beta} + 2t_{[\alpha\beta]} = 0. \tag{9}$$

The rotation coefficients and the tetrad have been treated here as independent fields in the variation. However, since we have chosen the rotation coefficients to be zero in the initial reference system and since we know how they transform,[3] we can easily show that they can be constructed from the tetrad. The infinitesimal form is

$$\zeta_{\alpha\beta\mu} = \partial_\mu\zeta_{[\alpha\beta]} + \partial_\alpha\zeta_{(\beta\mu)} - \partial_\beta\zeta_{(\alpha\mu)}. \tag{10}$$

Substituting this in (5) gives

$$\delta\mathfrak{L} \sim (t^{(\alpha\beta)} + \partial_\rho s^{(\alpha\beta)\rho})\zeta_{(\alpha\beta)} - (t^{[\alpha\beta]} + \tfrac{1}{2}\partial_\rho s^{\rho\alpha\beta})\zeta_{[\alpha\beta]}. \tag{11}$$

The second term vanishes on account of (9) and the first term can be reexpressed, using (9), as

$$\delta\mathfrak{L} \sim \theta^{\alpha\beta}\zeta_{(\alpha\beta)}, \tag{12}$$

where the symmetric tensor $\theta^{\alpha\beta}$ is

$$\theta^{\alpha\beta} = t^{\alpha\beta} + \partial_\rho(\tfrac{1}{2}s^{\rho\alpha\beta} - s^{(\alpha\beta)\rho}). \tag{13}$$

For *integral spin fields*, an alternative (and more usual) way of generalizing $L(\phi, \partial_\mu\phi)$ to curvilinear coordinates is by means of the metric and the Christoffel symbols, without the introduction of a tetrad:

$$\mathfrak{L}(\phi,\ \partial_\mu\phi,\ g_{\mu\nu},\ \partial_\rho g_{\mu\nu}). \tag{14}$$

Because (1) and (14) are equal in the initial (Cartesian) system, they are equal in any system, because they have the same transformation properties for coordinate changes and tetrad changes. The canonical SE tensor density is defined by arbitrary variation of the metric $\delta g_{\mu\nu} = \xi_{\mu\nu}$:

$$\delta\mathfrak{L} \sim \tfrac{1}{2} T^{\mu\nu}\xi_{\mu\nu} \tag{15}$$

so that

$$T_{\mu\nu} = 2\left(\partial_\rho\left(\frac{\partial\mathfrak{L}}{\partial\partial_\rho g^{\mu\nu}}\right) - \frac{\partial\mathfrak{L}}{\partial g^{\mu\nu}}\right). \tag{16}$$

For *half-integer* spin this tensor is undefined, because a spinor index is *essentially* related to tetrad rotations, not to coordinate transformations. The Lagrangian density of a half-integer spin field *necessarily* contains the tetrad components. However, if we are interested only in the reference systems that differ infinitesimally from Cartesian ones, the canonical SE tensor (16) can be defined for half-integer spin *provided* we destroy the independence of infinitesimal tetrad rotation and infinitesimal coordinate transformations. The simplest way of doing this is to impose the restriction

$$\zeta_{[\alpha\beta]} = 0 \tag{17}$$

on the tetrad variations. This condition is implicit in Goedecke's treatment of the Dirac field, though he does not explicitly introduce the tetrad concept.

Now, because of the orthonormality of the tetrad

$$\eta_{\alpha\beta} h^\alpha_\mu h^\beta_\nu = g_{\mu\nu} \tag{18}$$

we have

$$\xi_{\mu\nu} = 2\zeta_{(\mu\nu)}, \tag{19}$$

and (15) is

$$\delta\mathfrak{L} \sim T^{\mu\nu}\zeta_{(\mu\nu)}. \tag{20}$$

Comparison of (12) and (20) gives immediately, in the Cartesian system (by an integration and application of Gauss's theorem),

$$T_{\mu\nu} = \theta_{\mu\nu}. \tag{21}$$

[Incidentally, the identity $\partial_\mu T^{\mu\nu} = 0$ follows from substituting in (15) the variation $\xi_{\mu\nu} = -\partial_\mu\xi_\nu - \partial_\nu\xi_\mu$, integrating, and applying Gauss's theorem.]

Thus the equality of the Belinfante SE tensor and the canonical SE tensor is established if it can be shown that $t_{\mu\nu}$ is actually the Noether SE tensor, i.e., we have to show that

$$t^\mu_\nu = \delta^\mu_\nu\mathfrak{L} - \pi^\mu\partial_\nu\phi, \quad s^\mu{}_{\alpha\beta} = \pi^\mu G_{\alpha\beta}\phi, \tag{22}$$

where

$$\pi^\mu = \partial\mathfrak{L}/\partial\partial_\mu\phi. \tag{23}$$

These follow from the form (1) of $\mathfrak{L}$. Note that, in (1), $L(\phi,\ \phi_\alpha)$ is constructed only from $\phi,\ \phi_\alpha$ and the Minkowski metric; the tetrad components do not occur explicitly, but only in the structure of ϕ_α. For variation of the tetrad and rotation coefficients,

$$\begin{aligned}\delta\mathfrak{L} &= \delta h L + h\delta L = \delta h L + h(\partial L/\partial\phi_\alpha)\delta\phi_\alpha \\ &= \zeta^\mu_\mu\mathfrak{L} - (\partial\mathfrak{L}/\partial\phi_\alpha)(-\zeta^\mu_\alpha\phi_\mu + \tfrac{1}{2}\zeta^{\alpha\beta}{}_\mu G_{\alpha\beta}\phi) \\ &= (\delta^\mu_\nu\mathfrak{L} - \pi^\mu\phi_\nu)\zeta^\nu_\mu + \tfrac{1}{2}(\pi^\mu G_{\alpha\beta}\phi)\zeta^{\alpha\beta}{}_\mu.\end{aligned}$$

Comparing this expression with (5) identifies the tensors t^μ_ν and $s^\mu{}_{\alpha\beta}$, which coincide with (22) in the Cartesian system.

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[3] T.W.B. Kibble, J. Math. Phys. **2**, 212 (1961).
[4] D.W. Sciama, Proc. Camb. Phil. Soc. **54**, 72 (1958).
[5] D.W. Sciama, *Recent Developments in General Relativity* (Pergamon, London, and PWN, Warsaw, 1962).