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Conformal invariance and scalar-tensor theories

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Abstract. A new generalisation of Einstein's theory is proposed which is invariant under conformal mappings. Two scalar fields are introduced in addition to the metric tensor field, so that two special choices of gauge are available for physical interpretation, the 'Einstein gauge' and the 'atomic gauge'. The theory is not unique but contains two adjustable parameters ζ and α. With $\alpha = 1$ the theory viewed from the atomic gauge is Brans-Dicke theory ($\omega = -3/2 + \zeta/4$). Any other choice of α leads to a creation-field theory. In particular the theory given by the choice $\alpha = -3$ possesses a cosmological solution satisfying Dirac's 'large numbers' hypothesis.

1. Introduction

If we set $\hbar = c = 1$, every physical quantity Q has units which are a power N of a length. A conformal mapping is a spacetime-dependent change in the unit of length,

$$g_{\mu\nu} \rightarrow \lambda^2 g_{\mu\nu}$$
$$Q \rightarrow \lambda^N Q \tag{1.1}$$

The earliest attempt to make use of this kind of transformation in physics was made by Weyl (1919). In order to construct a conformally-invariant (co-invariant) Lagrangian density a dimensionless vector field ϕ_μ with the anomalous transformation law

$$\phi_\mu \rightarrow \phi_\mu + \partial_\mu \ln \lambda \tag{1.2}$$

is introduced. Weyl's Lagrangian density is

$$(-g)^{1/2} (P^2 - \tfrac{1}{4} \phi_{\mu\nu}\phi^{\mu\nu}) \tag{1.3}$$

where

$$P = R - 6 (\phi^\mu \phi_\mu - \phi^\mu_{|\mu}). \quad \phi_{\mu\nu} = \phi_{\mu.\nu} - \phi_{\nu.\mu}$$

(we denote partial differentiation by a dot and covariant differentiation constructed from Christoffel symbols by |). A generalised conformally covariant (co-covariant) derivative of any tensor field Q can be constructed by adding extra terms to the derivative. Co-covariant differentiation is denoted by $Q_{;\mu}$. The transformation law is $Q_{;\mu} \rightarrow \lambda^N Q_{;\mu}$ [for specific details of the construction, see Lord (1972) or Dirac (1973 *b*)]. Conformal invariance requires the square of the curvature scalar to occur in Weyl's Lagrangian, so the field equations for the metric are fourth order in Weyl's theory. The theory given by (1.3) was believed by Weyl to be a theory of gravitation and electromagnetism in free space. If matter is present the conformal invariance is necessarily broken because the matter Lagrangian will contain particle masses and coupling constants and of course the constancy of these quantities is not compatible with conformal invariance.

Dirac's conformally covariant theory (Dirac 1973 *b*) contains Weyl's vector and also a scalar field σ with conformal transformation law

$$\sigma \rightarrow \lambda^{-1} \sigma \tag{1.4}$$

Conformal invariance in the presence of matter is secured by replacing particle masses and the Newtonian gravitational constant by multiples of σ and σ^{-2} respectively. A conformally invariant Lagrangian density for σ is now required. Dirac's conformally invariant Lagrangian density (in the most general form of the theory) in the absence of matter is

$$(-g)^{1/2} (\sigma^2 R - \tfrac{1}{4} \phi_{\mu\nu}\phi^{\mu\nu} - k\sigma_{;\mu}\sigma^{;\mu}) \tag{1.5}$$

where

$$\sigma_{;\mu} = \sigma_{.\mu} + \phi_\mu \sigma$$

To the action associated with this Lagrangian density, a co-invariant action for matter is added.

Pietenpol, Incoul and Speiser (1974) have observed that the conformal invariance of Dirac's theory is only a formal mathematical property, devoid of physical significance. The argument put forward by these authors employed the specific form of Dirac's action functional, but in fact it applies to any conformally invariant theory which employs a *single scalar field* for the purpose of replacing coupling constants and particle masses to achieve conformal covariance. To see this we may simply note that by a conformal mapping we can set $\sigma = 1$, so that so far as physical interpretations are concerned such a theory is fully equivalent to one which lacks conformal covariance and in which coupling constants and masses are true constants. Any other choice of gauge is a mathematical abstraction, devoid of physical relevance.

2. Two-scalar theories

The conclusion to be drawn from the result of Pietenpol *et al* (1974) is that conformal invariance can be achieved in a physically non-trivial way only if we have (at least) *two* scalar fields to take the place of coupling constants and masses.

In a theory containing a scalar field σ, it is not strictly necessary to introduce a Weyl vector as an additional field for forming co-covariant derivatives. A Weyl vector can be constructed from σ by the prescription

$$\phi_\mu = -\sigma^{-1} \sigma_{.\mu} = -\partial_\mu (\ln \sigma) \tag{2.1}$$

Weyl and Dirac introduced a true vector (rather than the derivative of a scalar) because they wished to interpret it as the electromagnetic four-potential. However, we have shown elsewhere (Lord 1972) that, for the co-covariant derivative of a four-component spinor, $\gamma^\mu\psi_{;\mu} = \gamma^\mu\psi_{|\mu}$, so that Weyl's vector would not in fact couple to an electron. An independent Weyl vector different from (2.1) is required only if we wish to be able to construct a non-vanishing co-covariant derivative of σ. In the two-scalar theory this is not required so it will be sufficient to construct the Weyl vector according to (2.1). The co-covariant derivative of any quantity Q is then simply

$$Q_{;\mu} = \sigma^{-N} (\sigma^N Q)_{,\mu} \tag{2.2}$$

where the comma denotes a covariant derivative constructed from the Christoffel symbols associated with the dimensionless metric

$$\bar{g}_{\mu\nu} = \sigma^2 g_{\mu\nu} \tag{2.3}$$

If we make use of the dimensionless metric $\bar{g}_{\mu\nu}$, the obvious conformally invariant generalisation of Einstein's Lagrangian density is

$$(-\bar{g})^{1/2} \bar{R} = (-g)^{1/2} (\sigma^2 R + 6\sigma \Box \sigma) \tag{2.4}$$

where

$$\Box \sigma = (-g)^{-1/2} [(-g)^{1/2} g^{\mu\nu} \sigma_{,\mu}]_{,\nu}$$

($\bar{g}$ is the determinant of the dimensionless metric and $\bar{R}$ is the curvature scalar constructed from this 'metric'). The expression (2.4) differs only by a divergence from

$$(-g)^{1/2} (\sigma^2 R - 6\sigma_{,\mu}\sigma^{,\mu}). \tag{2.5}$$

A co-covariant Lagrangian density for a set of 'matter' fields Q can be obtained by replacing derivatives by the generalised derivatives (2.2) and by replacing masses by multiples of a scalar field, coupling constants by appropriate powers of a scalar field. If we use σ for constructing masses and coupling constants, the criticism of Pietenpol *et al* will apply. We must, therefore, introduce an additional scalar field κ for this purpose [A 'mass field' of this kind was first introduced by Hoyle and Narlikar (1971)]. Finally, we must construct a conformally invariant Lagrangian density for the mass field. An obvious choice is

$$\zeta (-\bar{g})^{1/2} \bar{g}^{\mu\nu} \kappa_{;\mu}\kappa_{;\nu}/\kappa^2 \tag{2.6}$$

where

$$\kappa_{;\mu} = \sigma (\kappa/\sigma)_{,\mu} \tag{2.7}$$

and ζ is a dimensionless parameter. The full proposed Lagrangian density is now

$$(-\bar{g})^{1/2} (\bar{R} + \zeta \bar{g}^{\mu\nu} \kappa_{;\mu}\kappa_{;\nu}/\kappa^2) + \mathcal{L} \tag{2.8}$$

where the matter Lagrangian $\mathcal{L}$ contains σ and κ.

Of course, we are not led uniquely to this form by the requirement of conformal invariance. We have been guided also by considerations of simplicity. The theory has two special gauges. In the gauge in which $\sigma = 1$, the gravitational 'constant' is a true constant. Since $\sigma^2 = 1/16\pi G$, we shall call this the Einstein gauge. In the gauge in which $\kappa = 1$, particle masses and coupling constants *other than G* are true constants; this may be called the atomic gauge. In the two gauges, the Lagrangian density becomes

$$(-g)^{1/2}(R + \zeta\kappa_{.\mu}\kappa^{.\mu}/\kappa^2) + \mathcal{L} \tag{2.9}$$

and

$$(-g)^{1/2}[\sigma^2 R + (-6+\zeta)\sigma_{.\mu}\sigma^{.\mu}] + \mathcal{L} \tag{2.10}$$

respectively.

Note that with the particular choice (2.6) we get the simplest possible Lagrangian in the atomic gauge [for instance the choice $(-g)^{-1/2}\kappa_{;\mu}\kappa^{;\mu}$ would have given rise to the peculiar form $(-g)^{1/2}(-6+\zeta\sigma^{-2})\sigma_{.\mu}\sigma^{.\mu}$ in the atomic gauge]. We have chosen the Weyl vector (2.1) rather than $-\partial_\mu(\ln\kappa)$ because otherwise we could not have constructed a co-covariant derivative of κ. In a previous approach to two-scalar theory (Lord 1974) a different, less transparent, notation was used, but the Lagrangian density suggested there was in fact equivalent to (2.8) but with a different scheme for making $\mathcal{L}$ conformally covariant [the reader should note that the inadvertant omission of σ in equation (2.7) of this reference led to spurious factors e^σ in (5.8) and subsequent equations. Section 6 of that paper is consequently wrong]. Other variations also exist. We could use κ as a mass field but construct coupling constants from σ or even $\sigma f(\kappa/\sigma)$ where f is an arbitrary function. The point is that the requirement of conformal invariance is quite inadequate for determining the precise way in which σ and κ should appear in a matter Lagrangian. In the following section we avoid these difficulties by considering only the case when the matter can be treated as a pressureless fluid, and supplementing the equations by a simple hypothesis about the sources of σ and κ.

A remarkable feature of co-covariant theories is that mass terms are replaced by interaction terms representing a coupling between Q and κ, with dimensionless coupling constant. All other coupling constants are also dimensionless. Thus, a quantisation of such a theory would require no mass and coupling constant renormalisations.

3. Pressureless fluid

If we carry out variations of $g_{\mu\nu}$, σ, κ and Q in a co-invariant matter Lagrangian density $\mathcal{L}$, we obtain

$$\delta\mathcal{L} \sim \tfrac{1}{2}\mathfrak{T}^{\mu\nu}\,\delta g_{\mu\nu} + \mathfrak{S}\delta\sigma + \mathfrak{K}\delta\kappa + \mathfrak{F}\,.\,\delta Q \tag{3.1}$$

where $\sim$ denotes that a divergence has been discarded. $\mathfrak{F}$ vanishes because $\mathfrak{F}=0$ is just the set of field equations for Q. The quantities $\mathfrak{T}^{\mu\nu}$, $\mathfrak{S}$ and $\mathfrak{K}$ are the 'sources' of $g_{\mu\nu}$, σ and κ. By taking the variations to be those brought about by an infinitesimal coordinate transformation, integrating (3.1) over a region of spacetime on the boundary of which the variations vanish, and applying Gauss' theorem, we obtain

$$\mathfrak{T}^{\nu}_{\mu|\nu} = \kappa_{.\mu}\mathfrak{K} + \sigma_{.\mu}\mathfrak{S} \tag{3.2}$$

By taking the variations to be those brought about by an infinitesimal conformal mapping, we get

$$\mathfrak{T} = \kappa\mathfrak{K} + \sigma\mathfrak{S} \tag{3.3}$$

For a pressureless fluid,

$$\mathcal{L} = -(-g)^{1/2}\rho, \quad T^{\mu\nu} = (-g)^{-1/2}\mathfrak{T}^{\mu\nu} - \rho u^{\mu} u^{\nu}, \tag{3.4}$$

but we have no prescription for determining $K = (-g)^{-1/2}\mathfrak{K}$ and $S = (-g)^{-1/2}\mathfrak{S}$. The theory is complete only when supplemented by an additional hypothesis about the form of these sources. We make the reasonable hypothesis that the *source of K is proportional to the density* ρ

$$K = \alpha\rho/\kappa \tag{3.5}$$

Here, α is a dimensionless parameter and we have included a factor $1/\kappa$ to make (3.5) dimensionally constant. Then (3.3) leads immediately to

$$S = (1 - \alpha)\,\rho/\sigma \tag{3.6}$$

Let the mass of an individual particle of the fluid be denoted by m and the number density by ν. Then

$$\rho = m\nu = \kappa\mu\nu \tag{3.7}$$

where μ is a dimensionless number. Define the dimensionless variable

$$f = \ln(\kappa^{\alpha}\sigma^{1-\alpha}) \tag{3.8}$$

Then (3.2) is just

$$T^{\mu\nu}_{\ |\nu} = \rho f^{\cdot\mu} \tag{3.9}$$

Substitute $T^{\mu\nu} = \rho u^{\mu} u^{\nu}$ and after some simple manipulations, making use of (3.7), eq. (3.9) is seen to imply

$$(\rho u^{\nu})_{|\nu} = \rho\, \mathrm{d}f/\mathrm{d}s \tag{3.10}$$

$$(\nu u^{\nu})_{|\nu} = \nu\frac{\mathrm{d}}{\mathrm{d}s}(f - \ln\kappa) \tag{3.11}$$

$$u^{\mu}u^{\nu}_{\ |\mu} = Q^{\nu}_{\mu} f^{\cdot\mu} \tag{3.12}$$

$\mathrm{d}/\mathrm{d}s = u^{\mu}\partial_{\mu}$ is differentiation along the world lines of the fluid, and

$$Q^{\nu}_{\mu} = \delta^{\nu}_{\mu} - u_{\mu}u^{\nu} \tag{3.13}$$

is the projection operator for projection orthogonal to the world lines. Equation (3.11) implies that, in general, *particle number is not conserved.* We have a theory in which the world lines have end-points, as in Hoyle-Narlikar theory.

We have a special case if $f = \ln\kappa$, *i.e.*, if $\alpha = 1$. With this choice of parameter we obtain a theory in which particle number is conserved. We have $S = 0$ and the resulting theory, in the atomic gauge, is simply Brans-Dicke theory [(2.11) with $\delta\mathcal{L}/\delta\sigma = 0$] with $\omega = -3/2 + \zeta/4$. (The Brans-Dicke field is $\phi = \sigma^2$—note that our R is the negative of the R of Brans and Dicke). In the Einstein gauge the energy-momentum tensor is not conserved, but this is not a theory with true creation since the non-conservation of energy comes from variation of the rest masses of individual particles rather than from creation of new particles: the property of world lines, of having no end-points, is clearly a conformally invariant property so is necessarily true in any gauge if true in one gauge. The co-covariance of the continuity equation $(\nu u^{\mu})_{|\mu} = 0$ is demonstrated by noting that it can be written $((-g)^{1/2}\nu u^{\mu})_{.\mu}$. Then since $(-g)^{1/2}$, ν and u^{μ} have respectively $N = 4, -3$ and -1, $(-g)^{1/2}\nu u^{\mu}$ is a co-invariant.

Thus, although Brans-Dicke theory is not conformally covariant it can be regarded as the aspect of a co-covariant theory given by a particular choice of gauge. This is illuminating in view of the existence of broken symmetries in physics. It could be that the underlying physical laws are completely invariant under a certain group of transformations, the symmetry being *apparently* broken only because a particular gauge is singled out by the assumptions implicit in the process of observations and measurements.

Note that free particles do not in general move on geodesics in our theories, because of the term on the right-hand side of (3.12). The form of (3.12) shows that f behaves in a sense like a pressure. There is in fact a third special gauge, determined by $f = 0$, in which the geodesic hypothesis and the conservation of energy hold.

4. Cosmological solutions

For convenience, we can work in the Einstein gauge ($\sigma = 1$) and look for solutions of the equations that follow from (2.10) and (3.5), of Robertson-Walker form

$$\left.\begin{aligned} ds^2 &= dt^2 - S^2(t)\, d\lambda^2, \\ d\lambda^2 &= (1 + kr^2/4)^{-2}\left(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right) \end{aligned}\right\} \tag{4.1}$$

$$\kappa = \kappa(t). \tag{4.2}$$

Having found a solution in this gauge, it is a simple matter to go over to the atomic gauge by carrying out a conformal transformation. The equation for κ is

$$2\zeta\left[\left(\frac{\dot\kappa}{\kappa}\right)^2 - \frac{\ddot\kappa + 3\frac{\dot S}{S}\dot\kappa}{\kappa}\right] = \alpha\rho \tag{4.3}$$

and, choosing a co-moving coordinate system $[u^\mu = (0, 0, 0, 1)]$, the modified Einstein equations give

$$\frac{k}{S^2} + \left(\frac{\dot S}{S}\right)^2 + 2\frac{\ddot S}{S} + \frac{\zeta}{2}\left(\frac{\dot\kappa}{\kappa}\right)^2 = 0 \tag{4.4}$$

$$\frac{3k}{S^2} + 3\left(\frac{\dot S}{S}\right)^2 - \frac{\zeta}{2}\left(\frac{\dot\kappa}{\kappa}\right)^2 = \frac{1}{2}\rho \tag{4.5}$$

We shall concern ourselves only with the special linearly expanding solutions,

$$S \sim t, \quad \kappa \sim t^n \ (n \neq 0). \tag{4.6}$$

Then (4.3) gives

$$\rho = (-4\zeta n/\alpha)\, t^{-2} \tag{4.7}$$

Substituting this in (4.5) and eliminating the term involving k from (4.4) and (4.5) gives

[Equations (4.8) and (4.7) incidentally imply that ζ must be negative in order for a physically relevant solution, of the kind we are seeking, to exist. Substituting in (4.4) we see that in fact, with $k = 1$ (closed universes), such a solution will exist only if $\zeta < -2\alpha^2$].

We now carry out a conformal mapping, to the atomic gauge, on the solution

$$\left.\begin{aligned} &\sigma = 1 \\ &\kappa \sim t^{1/\alpha} \\ &S \sim t \\ &\rho \sim t^{-2} \end{aligned}\right\} \qquad (4.9)$$

The transformation is

$$\kappa \to 1, \quad \sigma \to t^{-1/\alpha}, \quad \mathrm{d}s \to t^{1/\alpha}\,\mathrm{d}s, \quad \rho \to t^{-4/\alpha}\,\rho$$

If we change to a new time coordinate

$$\tau \sim t^{1+1/\alpha} \qquad (\text{for } \alpha \neq -1) \qquad (4.10)$$

the new line interval again has Robertson-Walker form, and we have, in the new gauge,

$$\left.\begin{aligned} &\kappa = 1 \\ &\sigma \sim \tau^{-1/(1+\alpha)} \\ &S \sim \tau \\ &\rho \sim \tau^{-(4+2\alpha)/(1+\alpha)} \end{aligned}\right\} \qquad (4.11)$$

A particularly interesting solution arises when $\alpha = -3$. The Newtonian gravitational 'constant' is $G \sim \sigma^{-2}$ so we have

$$\left.\begin{aligned} &G \sim \tau^{-1} \\ &S \sim \tau \\ &\rho \sim \tau^{-1} \end{aligned}\right\} \qquad (4.12)$$

and N, the number of particles in a volume expanding with the matter, satisfies $N \sim \rho S^3$. Thus we have a cosmology with

$$G \sim \tau^{-1}, \quad N \sim \tau^2. \qquad (4.13)$$

These are precisely the time-dependences required by Dirac's large numbers hypothesis (Dirac 1973 *a*). Thus Dirac cosmology is a solution of the theory with $\alpha = -3$.

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