GRAVITATIONAL THEORY IN THE PRESENCE OF SPINORS *

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ABSTRACT

The theory of spinor fields in interaction with gravitation is described. The massless Dirac equation is modified by the interaction and becomes identical in form with the nonlinear spinor equation that proves the basis of Heisenberg's Unified Field Theory. When applied to electromagnetism the theory leads to an unresolved difficulty connected with charge conservation and the spin-density of electromagnetism.

IN special relativity, the Lorentz transformations, interpreted as *rigid rotations of a Cartesian coordinate* system in flat specetime, are fundamental; the basic equations of physics are constructed in such a way that their form is independent of the choice of Cartesian coordinate system. That is they are required to be invariant under Lorentz transformations. Thus the most general admissible equation is a *spinor* equation.

In the presence of gravitation the introduction of a Cartesian reference system is no longer possible, we are forced from the outset to work with general (curvilinear) coordinates. The requirement of Lorentz invariance has to be abandoned in favour of general covariance. The representations of general coordinate transformations are tensor densities; the generalisation of a Lorentz-invariant tensor equation to a generally covariant equation is straightforward, and simply consists of assigning 'weights' to tensor field and replacing derivatives by covariant derivatives. However, there is no such procedure that can be applied to spinor equations. Thus, when Dirac discovered the special-relativistic equation for the electron, which is a spinor equation, the problem immediately arose of how do describe mathematically an electron in a gravitational field. Can we assign a meaning to a spinor when spacetime is not flat, and can we construct a generally covariant Dirac equation?

The solution given by Schrödinger was to regard a spinor as *invariant* for coordinated changes, and to regard the Lorentz group that is represented by Spinor transformations as the group of rotations of a set of orthonormal reference vector fields, not a coordinate transformation. Thus, choosing four orthonormal vectors $h_a{}^\mu$ $(a = 1, 4)$ at each spacetime point (a 'tetrad'),

$$h_a{}^\mu h_a{}^\nu g_{\mu\nu} = \eta_{ab},\ \eta_{ab} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

and

$$h_\mu{}^a h_\nu{}^b \eta_{ab} = g_{\mu\nu}$$

where $h_\mu{}^a$ is the inverse of the matrix $h_a{}^\mu$. Change to a new tetrad is given by

$$\bar{h}_a{}^\mu = \Lambda_a{}^b h_b{}^\mu$$

where $\Lambda_a{}^b$ is a (spacetime-dependent) Lorentz matrix. Let γ^a be the four usual Dirac matrices. Then $\gamma^\mu = h_a{}^\mu \gamma^a$ satisfy

$$\tfrac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g^{\mu\nu}$$

and the spinor representation L of the Lorentz matrix Λ is given by

$$L^{-1} \gamma^a L = \Lambda_b{}^a \gamma^b.$$

The derivative $\partial_\mu \psi$ of a spinor transforms under tetrad rotation to

$$L\partial_\mu \psi + (\partial_\mu L)\psi$$

so that a spinor connection Γ_μ has to be introduced, with transformation law defined so that the 'covariant' derivative $\psi_{;\mu} = \partial_\mu \psi - \Gamma_\mu \psi$ transforms simply to $L\psi_{;\mu}$. The equation $\gamma^\mu \psi_{;\mu} + m\psi = 0$ is

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then invariant under both general coordinate transformations and Lorentz rotations of that tetrad, and is the required generalisation of the Dirac equation.

The 'spin coefficients' or 'Ricci rotation coefficients'

$$\lambda^{\mu}{}_{a\rho} = h^{\mu}{}_{a|\rho} = \partial_{\rho} h_a{}^{\mu} + \Gamma^{\mu}{}_{\lambda\rho} h_a{}^{\lambda}$$

possesses the skewsymmetry $\lambda_{\rho}{}^{ab} = -\lambda_{\rho}{}^{ba}$. A spinor connection can be constructed from the spin coefficients according to $\Gamma_{\rho} = -\frac{1}{4}\lambda_{ab\rho}\gamma^a\gamma^b$. The components of such a spinor connection, determined entirely by the $h_a{}^{\mu}$ and their derivatives, are the Fock-Ivanenko coefficients.

The above prescription appears at first sight to be a complete solution to the problem of spinors in general relativity. This is not so; the work of Kibble[1] and Sciama[2] leads to the conclusion that the existence of spinors requires a profound modification of Einstein's gravitational theory.

Two facts about Einstein's theory are significant. First, the energy-momentum tensor on the right-hand side of Einstein's field equations is necessarily a symmetric tensor. The *canonical* energy momentum tensors of various physical fields that arise from consideration of Noether's theorem in special relativity in general have a skewsymmetric part. Second, the field equations are obtained by variation of the Lagrangian

$$\frac{1}{2k}(-g)^{\frac{1}{2}} R + \mathcal{L}$$

with respect to $g_{\mu\nu}$.

When $\mathcal{L}$ is the Lagrangian density

$$(-g)^{\frac{1}{2}}(\bar{\psi}\gamma^{\mu}\psi_{,\mu} - \bar{\psi}_{,\mu}\gamma^{\mu}\psi)$$

of the Dirac field this variational principle is not properly defined, since this scalar density contains the 16 fields $h_a{}^{\mu}$ instead of the 10 $g_{\mu\nu}$. Thus it seems that the $h_a{}^{\mu}$ rather than the $g_{\mu\nu}$ ought to be regarded as the field variables for the gravitational field. This possibility has been investigated by various authors (Sciama[2], Møller[3], Pelegrini and Plebanski[4]).

Following Sciama, we use an approach which is analogous to Palatini's approach to the variational principle of conventional general relativity. We express

$$\frac{1}{2k}(-g)^{\frac{1}{2}} R + \mathcal{L}$$

in terms of $h_a{}^{\mu}$ and $\lambda^{ab}{}_{\mu}$ and treat $h_a{}^{\mu}$ and $\lambda^{ab}{}_{\mu}$ as independent variables in the variation.

The Riemann tensor is

$$R^{ab}{}_{\rho\sigma} = \lambda^{ab}{}_{\rho,\sigma} - \lambda^{ab}{}_{\sigma,\rho} - \lambda_{\sigma}{}^{cb}\lambda_{c\rho}{}^{a} + \lambda_{\rho}{}^{cb}\lambda^{a}{}_{c\sigma}$$

so that

$$(-g)^{\frac{1}{2}} R \text{ is}$$

$$-h^{\rho\sigma}{}_{ab}(\lambda^{ab}{}_{\rho,\sigma} + \lambda^{cb}\lambda^{a}{}_{c\sigma\rho})$$

where

$$h^{\rho\sigma}{}_{ab} = h\,(h_a{}^{\rho} h_b{}^{\sigma} - h_a{}^{\sigma} h_b{}^{\rho}),$$

$$h = \det(h_{\mu}{}^{a}) = (-g)^{\frac{1}{2}}.$$

Variation of $h_a{}^{\mu}$ gives

$$h\,(R_{\mu}{}^{a} - \tfrac{1}{2} h_{\mu}{}^{a} R) + k\tau_{\mu}{}^{a} = 0$$

where the tensor density $\tau_{\mu}{}^{a}$ is given by

$$\mathcal{L} = \delta\tau_{\mu}{}^{a}\delta h_a{}^{\mu}.$$

Thus we obtain 16 equations instead of the usual 10:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + k T_{\mu\nu} = 0$$

where $R_{\mu\nu}$ and $T_{\mu\nu}$ are not necessarily symmetric.

Variation of $\lambda^{ab}{}_{\rho}$ leads to

$$h\,(\Omega_{\mu\nu}{}^{\rho} + \delta_{\mu}{}^{\rho}\Omega_{\nu} - \delta_{\nu}{}^{\rho}\Omega_{\mu}) = k\mathfrak{S}_{\mu\nu}{}^{\rho}$$

where $\Omega_{\mu\nu}{}^{\rho}$ is the torsion tensor

$$\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu},\ \Omega_{\mu} = \Omega_{\mu\nu}{}^{\nu}$$

and $\mathfrak{S}_{\mu\nu}{}^{\rho}$ is given by

$$\delta\mathcal{L} = \tfrac{1}{2}\mathfrak{S}_{ab}{}^{\rho}\delta\lambda^{ab}{}_{\rho}$$

and is just the 'spin-density'; that is, the 'current' associated with the group of tetrad rotations.

For the Dirac equation (for simplicity omitting the mass term), the Lagrangian density

$$\mathcal{L} = h\,\tfrac{1}{2}(\bar{\psi}\gamma^{\mu}\psi_{|\mu} - \bar{\psi}_{|\mu}\gamma^{\mu}\psi)$$

$$= h\,[\tfrac{1}{2}(\bar{\psi}\gamma^{\mu}\psi_{,\mu} - \bar{\psi}_{,\mu}\gamma^{\mu}\psi) - \tfrac{1}{2}\bar{\psi}\{\Gamma_{\mu}, \gamma^{\mu}\}\psi]$$

contains the 'spin-coefficients' in the term

$$h\,.\,\tfrac{1}{8}\lambda^{\nu\rho\mu}\bar{\psi}\{\gamma_{\nu\rho}, \gamma_{\mu}\}\psi = \tfrac{1}{2} i\epsilon^{\nu\rho\mu\sigma}\bar{\psi}\gamma_{\sigma}\gamma_5\bar{\psi}\,.\,\tfrac{1}{2}\lambda_{\nu\rho\mu}.$$

Thus in this case, $\mathfrak{S}^{\nu\rho\mu}$ is completely skewsymmetric and its 'dual' is the axial vector

$$S_{\sigma} = \tfrac{1}{2}\bar{\psi}\gamma_{\sigma}\gamma_5\psi.$$

This introduces a torsion

$$\Omega_{\mu\nu\rho} = -kS_{\mu\nu\rho}.$$

The connection coefficients are found from

$$g_{\mu\nu/\rho} = 0,$$

$$\Gamma_{\mu\nu}{}^{\rho} = \{\rho_{\mu\nu}\} + \tfrac{1}{2}\Omega_{\mu\nu}{}^{\rho}$$

so that

$$\lambda^{\rho}{}_{a\nu} = (h^{\rho}{}_{a,\nu} + \{\rho_{a\nu}\}h_a{}^{a}) + \tfrac{1}{2}\Omega_{a\nu}{}^{\rho}.$$

Changing the notation slightly, using $\psi_{|\rho}$ to denote covariant differentiation using christoffel symbols (*i.e.*, covariant differentiation using Fock-Ivanenko coefficients constructed only from the first term on the right-hand side of the above expression), we can write

$$\mathcal{L} = h\,\tfrac{1}{2}(\bar{\psi}\gamma^{\mu}\psi_{|\mu} - \bar{\psi}_{|\mu}\gamma^{\mu}\psi) + \tfrac{1}{4}\Omega^{ab\rho}\mathfrak{S}_{ab\rho}.$$

The field equations become

$$0 = \gamma^{\mu}\psi_{|\mu} + \tfrac{1}{8}\Omega_{\mu\nu\rho} i\epsilon^{\mu\nu\rho\sigma}\gamma_{\sigma}\gamma_5\psi$$

$$= \gamma^{\mu}\psi_{|\mu} - \frac{k}{8}\mathfrak{S}_{\mu\nu\rho} i\epsilon^{\mu\nu\rho\sigma}\gamma_{\sigma}\gamma_5\psi$$

$$= \gamma^{\mu}\gamma_{|\mu} + (6k/8)\, S^{\sigma}\gamma_{\sigma}\gamma_5\psi.$$

That is, in the presence of gravitation the massless Dirac equation is expected to take the form

$$\gamma^{\mu}\psi_{|\mu} + (3k/8)\,\gamma^{\sigma}\gamma^{5}\psi\,.\,\bar{\psi}\gamma_{\sigma}\gamma_{5}\psi = 0.$$

It is remarkable that this has precisely the same form as the nonlinear spinor equation suggested by Heisenberg as the basis of his unified theory of elementary particles[5]. $3\kappa/8$ is Heisenberg's l_0^2 (thus $l_0 \sim 3 \times 10^{-33}$ cm).

For a vector field (omitting the mass term),

$$\mathcal{L} = -\tfrac{1}{4}\, h g^{\mu\nu} g^{\rho\sigma} f_{\mu\rho} f_{\nu\sigma}$$

where

$$f_{\mu\nu} = A_{\mu/\nu} - A_{\nu/\mu}$$
$$= A_{a\cdot\nu} h_{\mu}{}^{a} - A_{a\cdot\mu} h_{a}{}^{\nu} + A_{a}\,(\lambda_{\mu}{}^{a}{}_{\nu} - \lambda_{\nu}{}^{a}{}_{\mu}).$$

From the terms containing spin coefficients it is easy to pick out the spin tensor

$$\mathfrak{S}_{\mu\nu\rho} = f_{\mu\rho} A_{\nu} - f_{\nu\rho} A_{\mu}.$$

This is exactly the form of the spin tensor for electromagnetism deduced from the application of Noether's theorem in special relativity. However, this is not satisfactory, since the tensor $f_{\mu\nu}$ constructed here is not charge invariant. Thus in such a theory charge would not be conserved. The only way to retain charge conservation seems to be to construct the $f_{\mu\nu}$ in the Lagrangian from ordinary derivatives, in the usual way. But this gives a zero spin tensor for the spin-1 Maxwell field. Thus we have an unresolved anomaly when we try to formulate electromagnetism in the generalised theory.

The major unsolved problem in any theory which treats the Lorentz transformations as tetrad rotations rather than as coordinate transformations lies in the fact that we lose the fundamentally important Poincaré group (Inhomogeneous Lorentz group). The invariance group of the above generalised theory is a direct product of the (space-time dependent) homogeneous Lorentz group and the general coordinate transformation group. The invariance group of special relativity is the semi-direct product of the Lorentz group and the abelian translation group. Thus it is difficult to see how such a theory can be reconciled with special relativity— how the Poincaré—invariant formulation of particle physics can be regarded in some sense as a limiting case of a curved spacetime theory with a tetrad field.

Perhaps the most remarkable aspect of the generalised theory of Sciama is that the $h_a{}^{\mu}$, which is an aspect of the *reference system*, has been treated as a *dynamical field*. Nevertheless, no restriction on the tetrad is implied by this procedure (this is to be contrasted with the work of Møller, in which only tetrad fields satisfying a set of differential equations are admissible reference systems). The sharp distinction between the physics and the system of reference used to facilitate the description of the physics has been abandoned. This is true to a lesser extent in conventional general relativity: The physical fields $g_{\mu\nu}$ contain information about the coordinate system as well as information about the geometry, and the two kinds of information are quite incapable of separation in any given set of ten functions $g_{\mu\nu}$.

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