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Redshift Formulae for De-Sitter Type Metrics

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Abstract

Red-shift formulae are obtained for the three Robertson–Walker solutions of $R_{\mu\nu} = \Lambda g_{\mu\nu}$ by methods based on the geometry of a hyperquadric in five-space. The possibility that the solutions may have validity for the real universe is justified by an appeal to Hoyle–Narlikar theory.

1. *Introduction*

A previous paper (Lord, 1973) dealt with the geometrical aspects of the three solutions

$$dt^2 - \cosh^2 t(d\rho^2 + \sin^2\rho\, d\Omega^2) \qquad (k = +1) \tag{1.1}$$

$$dt^2 - e^{2t}(d\rho^2 + \rho^2\, d\Omega^2) \qquad (k = 0) \tag{1.2}$$

$$dt^2 - \sinh^2 t(d\rho^2 + \sinh^2\rho\, d\Omega^2) \qquad (k = -1) \tag{1.3}$$

of the equations

$$R_{\mu\nu} = 3g_{\mu\nu} \tag{1.4}$$

(The units of length and time have been chosen so that the velocity of light $c = 1$, and the cosmological constant $\Lambda = 3$.) It was shown that the three metrics correspond to different choices of coordinate system on a hyperquadric

$$r^2 - {\eta_4}^2 + {\eta_5}^2 = {\eta_1}^2 + {\eta_2}^2 + {\eta_3}^2 - {\eta_4}^2 + {\eta_5}^2 = 1 \tag{1.5}$$

in a flat five-space with signature (+++−+). The variables (t, ρ) are related to (r, η_4, η_5) through

$$\tan\rho = r/\eta_5; \qquad \sinh t = \eta_4 \qquad (k = +1) \tag{1.6}$$

$$\rho = r/(\eta_4 + \eta_5); \quad e^t = \eta_4 + \eta_5 \quad (k = 0) \tag{1.7}$$

$$\tanh\rho = r/\eta_4; \qquad \cosh t = \eta_5 \qquad (k = -1) \tag{1.8}$$

In the present work we shall relate these geometrical quantities to the observable quantities luminosity distance and red-shift.

There are two possible physical interpretations of the cosmological models given by (1.1)–(1.2):

(i) In the context of Einstein's theory, with zero pressure and mass density μ satisfying $(8\pi G\mu \ll \Lambda$, the cosmological (Robertson–Walker) solutions will approximate to (1.1)–(1.3). The quantity μS^3 (with $S = \cosh t$, e^t, $\sinh t$ respectively for $k = +1, 0, -1$) will be a constant. The inequality $(8\pi G\mu \ll \Lambda$, however, appears to be extremely unlikely on the basis of observational data: taking $(3/\Lambda)^{1/2}$ to be the Hubble constant ($\sim 10^{10}$ yr) and $\mu \sim 5 \times 10^{-29}$ gm.cm^{-3}, then (with units chosen so that $c = 1$) we get 10^{-57} cm^{-2} for both Λ and $8\pi G\mu$. Nothing in Einstein's theory excludes the possibility of matter with negative μ (i.e. matter giving rise to a repulsive gravitational field). A considerable amount of such matter in the form of unobservable inter-galactic dust could make the value of μ less than the observable value, or even zero. Although such a possibility cannot be ruled out on physical grounds, it nevertheless appears far-fetched if its only justification is to simplify the Einstein equation to (1.4) in the cosmological context. The strongest argument against it is of course that the equality in order of magnitude of Λ and $8\pi G\mu$ is too striking to be simply dismissed, particularly in view of the fact that several models can account for the coincidence theoretically.

(ii) The Hoyle–Narlikar creation-field cosmology (Hoyle, 1963) contains evolving models of the universe with constant μ (maintained by creation or destruction of matter). In the case of constant μ the C-field theory reduces to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\Lambda = 4\pi G\mu$. Thus, in this theory, the metrics (1.1)–(1.3) represent universes with a constant mass density given (in the units we are employing) by

$$\mu = \frac{3}{4\pi G} \tag{1.9}$$

If we postulate the three alternative cosmological models (1.1)–(1.3), the question of whether they arise from Einstein's theory (i) or Hoyle–Narlikar theory (ii), makes no difference to the expressions for red-shift and luminosity distance to be derived.

2. *Red-Shift*

For light leaving a galaxy at event $P(\rho, t)$ (equivalently (r, η_4, η_5)) and arriving at the observer at the spatial origin $Q(0, t_0)$ (equivalently $(0, \mathring{\eta}_4, \mathring{\eta}_5)$) the red-shift z is given by

$$\frac{dt_0}{dt} = 1 + z = \frac{S(t_0)}{S(t)} \tag{2.1}$$

where $S(t)$ is the function appearing in the Robertson–Walker metric ($\cosh t$, e^t and $\sinh t$ respectively for $k = +1, 0, -1$).

The emitting event (r,η_4,η_5) and the receiving event $(0,\mathring{\eta}_4,\mathring{\eta}_5)$ are on the same null geodesic. From the geometrical discussion in Lord (1973) we know that the events on the null geodesics through $(0,0,1)$ are of the form $(\alpha,\alpha,1)$ or $(\alpha,-\alpha,1)$. That is, (r,η_4,η_5) and $(\mathring{r},\mathring{\eta}_4,\mathring{\eta}_5)=(0,0,1)$ are on the same null geodesic and if only $\mathring{r}r-\mathring{\eta}_4\eta_4+\mathring{\eta}_5\eta_5=1$. Since this equation is invariant under $SO(2,1)$ rotations it represents the condition for any two points $(\mathring{r},\mathring{\eta}_4,\mathring{\eta}_5)$ and (r,η_4,η_5) to lie on a null geodesic. If we let the observer be at the spatial origin $(r=0)$,

$$r^2-\eta_4{}^2+\eta_5{}^2=1 \tag{2.2}$$

$$-\mathring{\eta}_4{}^2+\mathring{\eta}_5{}^2=1 \tag{2.3}$$

$$-\mathring{\eta}_4\eta_4+\mathring{\eta}_5\eta_5=1 \tag{2.4}$$

are satisfied by the parameters of the emitter and observer.

The red-shifts expressed in terms of the parameters, for the three models, are

$$k=+1:\quad 1+z=\frac{\operatorname{ch}t_0}{\operatorname{ch}t}=\sqrt{\left(\frac{\mathring{\eta}_4{}^2+1}{\eta_4{}^2+1}\right)}=\frac{\mathring{\eta}_5}{\sqrt{(r^2+\eta_5{}^2)}}$$

$$=\frac{\mathring{\eta}_5/\eta_5}{\sqrt{[(r/\eta_5)^2+1]}}=\frac{\mathring{\eta}_5}{\eta_5}\cos\rho \tag{2.5}$$

$$k=0:\quad 1+z=e^{t_0-t}=\frac{\mathring{\eta}_4+\mathring{\eta}_5}{\eta_4+\eta_5} \tag{2.6}$$

$$k=-1:\quad 1+z=\frac{\operatorname{sh}t_0}{\operatorname{ch}t}=\sqrt{\frac{\mathring{\eta}_5{}^2-1}{\eta_5-1}}=\frac{\mathring{\eta}_4}{\sqrt{(\eta_4{}^2-r^2)}}$$

$$=\frac{\mathring{\eta}_4/\eta_4}{\sqrt{[1-(r/\eta_4)^2]}}=\frac{\mathring{\eta}_4}{\eta_4}\cosh\rho \tag{2.7}$$

From (2.2)–(2.4) we easily find

$$\eta_4=\mathring{\eta}_4-r\mathring{\eta}_5 \tag{2.8}$$

$$\eta_5=\mathring{\eta}_5-r\mathring{\eta}_4 \tag{2.9}$$

so the right-hand sides of (2.4)–(2.7) can be written respectively as

$$\frac{\cos\rho}{1-r\mathring{\eta}_4/\mathring{\eta}_5},\qquad \frac{1}{1-r},\qquad \frac{\cosh\rho}{1-r\mathring{\eta}_5/\mathring{\eta}_4} \tag{2.10}$$

For $k=+1$ we have $\eta_4=\sinh t$ so $\mathring{\eta}_4/\mathring{\eta}_5=\mathring{\eta}_4/(1+\mathring{\eta}_4{}^2)^{1/2}=\tanh t_0$ and for $k=-1$, $\eta_5=\cosh t$ so $\mathring{\eta}_5/\mathring{\eta}_4=\mathring{\eta}_5/(\mathring{\eta}_5{}^2-1)^{1/2}=\coth t_0$

$$k=+1:\quad 1+z=\cos\rho/(1-r\tanh t_0) \tag{2.11}$$

$$k=0:\quad 1+z=1/(1-r) \tag{2.12}$$

$$k=-1:\quad 1+z=\cosh\rho/(1-r\coth t_0) \tag{2.13}$$

Of course, the radial parameters ρ and r are not observable. We require relations between $1 + z$ and luminosity distance λ.

3. *Luminosity Distance*

Light from $P(0,\eta_4,\eta_5)$ arriving at $Q(\mathring{r},\mathring{\eta}_4,\mathring{\eta}_5)$ will be distributed over a sphere of radius $|\mathring{r}|$. The energy per unit time received will be proportional to $|\mathring{r}|^{-2}$ and will also be diminished by a factor $(1+z)^{-1} = dt/dt_0$. The 'luminosity distance' measured by assuming the light falls off according to an inverse square law is therefore $(1+z)^{1/2}|r_0|$. This will also be the luminosity distance for a source P' at (r',η_4',η_5') and an observer Q' at $(\mathring{r},\mathring{\eta}_4',\mathring{\eta}_5')$, provided the coordinates of P' and Q' are connected to those of P and Q by an $SO(2,1)$ rotation in (r,η_4,η_5)-space that does not change the shape of the (ρ,t) coordinate net. For $k=-1$ such a transformation is rotation about the η_5-axis

$$\begin{pmatrix} r' \\ \eta_4' \\ \eta_5' \end{pmatrix} = \begin{pmatrix} \operatorname{ch}\chi & \operatorname{sh}\chi & . \\ \operatorname{sh}\chi & \operatorname{ch}\chi & . \\ . & . & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \eta_4 \\ \eta_5 \end{pmatrix} \tag{3.1}$$

and similarly for Q and Q'. Hence

$$\operatorname{ch}\chi . r' - \operatorname{sh}\chi . \eta_4' = 0$$

Therefore

$$\tanh\chi = r'/\eta_4' \tag{3.2}$$

and

$$\mathring{r} = \operatorname{ch}\chi . \mathring{r}' - \operatorname{sh}\chi . \mathring{\eta}_4' = \frac{1}{\sqrt{(\eta_4'^2 - r'^2)}}(\eta_4' r_0' - r' \eta_4') \tag{3.3}$$

Whence, the luminosity distance λ for a source at (r,η_4,η_5) and an observer at $(0,\mathring{\eta}_4,\mathring{\eta}_5)$ is given by

$$(k=-1): \quad \lambda = (1+z)^{1/2} \frac{r\mathring{\eta}_4}{\sqrt{(\eta_4{}^2 - r^2)}}$$

$$= (1+z)^{1/2} r \sqrt{\left(\frac{\mathring{\eta}_5 - 1}{\eta_5{}^2 - 1}\right)} \tag{3.4}$$

For $k=+1$ and $k=0$ the relevant $SO(2,1)$ rotations are respectively

$$\begin{pmatrix} \cos\theta & . & \sin\theta \\ . & 1 & . \\ -\sin\theta & . & \cos\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \alpha & \alpha \\ -\alpha & 1-\dfrac{\alpha^2}{2} & -\dfrac{\alpha^2}{2} \\ \alpha & \dfrac{\alpha^2}{2} & 1+\dfrac{\alpha^2}{2} \end{pmatrix}$$

in the above argument. We then obtain the expressions

$$(k=+1) \qquad \lambda=(1+z)^{1/2}\,r\,\sqrt{\left(\frac{\mathring{\eta}_4{}^2+1}{\mathring{\eta}_4{}^2+1}\right)} \tag{3.5}$$

and

$$(k=0) \qquad \lambda=(1+z)^{1/2}\,r\left(\frac{\mathring{\eta}_4+\mathring{\eta}_5}{\eta_4+\eta_5}\right) \tag{3.6}$$

Now, for $k=+1$ we have $\eta_4=\sinh t$ so the square root in (3.5) is just $\cosh t_0/\cosh t=(1+z)$. For $k=0$, $\eta_4+\eta_5=\mathrm{e}^t$ so the term in brackets in (3.6) is $\mathrm{e}^t/\mathrm{e}^{t_0}=1+z$, and for $k=-1$, $\eta_5=\cosh t$ so the square root in (3.4) is $\sinh t_0/\sinh t=1+z$. Thus, *in all three cases*, we have the relation

$$\lambda=(1+z)^{3/2}\,r \tag{3.7}$$

4. *Relation Between z and λ*

We now have sufficient information to obtain the relationship between luminosity distance and red-shift.

(i) $k=+1$. From $\tan\rho=r/\eta_5$, $\sinh t=\eta_4$ we can obtain $\cosh t_0=\mathring{\eta}_5$ and therefore

$$\tan\rho=\frac{r}{\eta_5}=\frac{r}{\mathring{\eta}_5}=\left(\frac{\mathring{\eta}_5}{\eta_5}\right)=\frac{r}{\cosh t_0}\,\frac{(1+z)}{\cos\rho} \quad \text{(using (2.5))}$$

so that $\sin\rho=r(1+z)\operatorname{sech}t_0$. Combining this with (2.11) we obtain

$$\sin\rho=r(1+z)\operatorname{sech}t_0 \tag{4.1}$$

$$\cos\rho=(1+z)(1-r\tanh t_0) \tag{4.2}$$

Hence

$$1=(1+z)^2(r^2-2\chi_0 r+1) \tag{4.3}$$

where

$$\chi_0=\tanh t_0 \tag{4.4}$$

Solving for r and substituting in (3.7) gives

$$\lambda=(1+z)^{3/2}\{\chi_0-\sqrt{[\chi_0{}^2-1+(1+z)^{-2}]}\} \tag{4.5}$$

(ii) $k=0$. In the context of Hoyle–Narlikar theory with constant density this is just the well-known ‘steady-state universe’. From (2.12) we have just $r=z/(1+z)$ so that

$$\lambda=z(1+z)^{1/2} \tag{4.6}$$

which is just (4.5) with $\chi_0=1$.

(iii) $k = -1$. The two equations analogous to (4.1) and (4.2) are

$$\sinh \rho = r(1+z)\,\mathrm{cosech}\, t_0 \tag{4.7}$$

$$\cosh \rho = (1+z)(1 - r \coth t_0) \tag{4.8}$$

and we easily arrive at (4.5) with

$$\chi_0 = \coth t_0 \tag{4.9}$$

5. *Discussion of the Three Models*

For small values of λ and z, (4.5) can be written as a power series

$$\lambda = \chi_0^{-1} z(1 + az + bz^2 + \cdots) \tag{5.1}$$

where

$$a = \tfrac{1}{2}\chi_0^{-2}, \qquad b = \tfrac{1}{8}(4\chi_0^{-4} - 6\chi_0^{-2} + 1) \tag{5.2}$$

We have chosen the units so that $\Lambda = 3$. In any other units luminosity distance is $\lambda(3/\Lambda)^{1/2}$. The first term in (5.1) therefore determines the Hubble

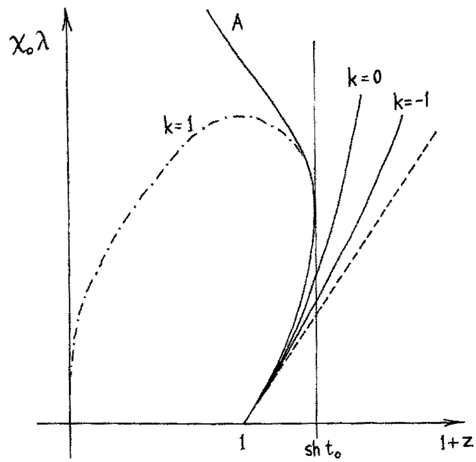

Figure 1. Luminosity distance versus redshift for de-Sitter metrics,

constant $\chi_0\sqrt{(\Lambda/3)}$. The second term determines χ_0, which is less than, equal to or greater than unity according as k is $+1$, 0 or -1. This determines the present epoch t_0. The third term is then a check on the validity of the models based on (1.4). For large z the expansion is inadequate. The full expression (4.5) is plotted qualitatively in Fig. 1. The dotted line corresponds to Hubble's law. For $k = +1$ the red-shift reaches a maximum and then starts to decrease as we go to fainter galaxies. The broken part of the curve corresponds to the contracting phase and is drawn on the assumption that the luminosities of galaxies are constant in time. The curve turns over, corresponding to apparent brightness increasing with increasing distance. This leads to Olber's paradox unless we postulate a non-luminous contraction phase. In a model consisting of non-luminous dust contracting, condensing into galaxies and then expanding, the broken part of the curve will be changed to a curve indicated qualitatively by A.

References

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