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Clifford Algebras and Representations of Complex Orthogonal Groups

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The 2^{ν}-dimensional spinor representations of the complex orthogonal group $SO(M, C)$ $(M = 2\nu + 2)$ are discussed. By making use of the possibility of regarding the elements of the Clifford algebra $C_{2\nu}$ as sets of skewsymmetric tensors in M dimensions (rather than in 2ν dimensions as in the usual treatment) general relations are obtained for the correspondences between rank 2 spinors and sets of tensors, valid for any dimensionality. The theory of two- and four-component spinors is discussed from the point of view obtained by considering the Lorentz group with reflections as a subgroup of $SO(6, C)$.

1. INTRODUCTION

The relationships that exist between the quaternion or Pauli algebra (C_2), the Dirac algebra (C_4), and the representations of the Lorentz group are well known on account of their fundamental importance to the calculus of two-component and four-component spinors. The aim of the present work is to demonstrate exactly how some of the well-known algebraic properties of C_2 and C_4 and the associated representations of the complex orthogonal group in four dimensions fit into the general scheme of the higher dimensional Clifford algebras. Of course, much work has already been done in this direction; in particular, the classic work of Brauer and Weyl [2], where the existence of the 2^{ν}-dimensional irreducible "spinor" representation of the complex orthogonal group $O(2\nu + 1, C)$ and its construction by means of the Clifford algebra $C_{2\nu}$ are demonstrated. Brauer and Weyl also point out that if the rotations $O(2\nu + 1, C)$ are restricted to the unimodular subgroup $SO(2\nu, C)$ then the 2^{ν}-dimensional spinor representation decomposes to give two inequivalent irreducible representations of dimension $2^{\nu-1}$. Our point of departure is the recognition of the fact that if we substitute $\nu + 1$ for ν in the above statement, it becomes a generalization of the original concept: There exist two inequivalent irreducible representations, of dimension 2^{ν}, of the group $SO(2^{\nu} + 2, C)$. Then $O(2^{\nu} + 1, C)$ is a subgroup of $SO(2\nu + 2, C)$,

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for which the two representations become equivalent, and we arrive again at the single 2^ν-dimensional irreducible representation of $O(2\nu + 1, C)$.

In order to deal algebraically with the 2^ν-dimensional representations of $SO(2\nu + 2, C)$ by means of the Clifford algebra $C_{2\nu}$, we shall treat the components of an element of $C_{2\nu}$ as a set of tensors in $2\nu + 2$ dimensions, rather than in 2ν dimensions, as in the usual treatment. In order to do this we are led to a generalization of the concept of "quaternion conjugation" to all even Clifford algebras.

2. Sets of Skewsymmetric Tensors

The basis for the Clifford algebra $C_{2\nu}$ can be chosen so that the components of a general element can be regarded as a set of skewsymmetric tensors of rank p ("p-forms"), one of each rank from $p = 0$ to $p = 2\nu$. We shall obtain other choices of the basis so that these components can be regarded as sets of skewsymmetric tensors in $2\nu + 1$ dimensions, or in $2\nu + 2$ dimensions. It will therefore be useful in clarifying the ideas if we set up for a formalism for discussing sets of p-forms in an n-dimensional space.

Let

$$a,\ a_\alpha\ ,\ a_{\alpha\beta\gamma}\ ,\dots,\ a_{\alpha_1\cdots\alpha_n} \tag{2.1}$$

be a set of complex quantities completely skewsymmetric in their n-fold indices. We will use the notation

$$\mathscr{A} = [a_{\alpha_1\cdots\alpha_p}] \tag{2.2}$$

to denote such a set. With the obvious definitions for addition and multiplication by a complex number the quantities $\mathscr{A}$ can be regarded as vectors in a vector space $\mathscr{V}(n)$ of dimension

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

A scalar product will be defined on $\mathscr{V}(n)$:

$$\mathscr{A} \cdot \mathscr{B} = \mathscr{B} \cdot \mathscr{A} = \sum_{p=0}^{n} \frac{1}{p!}\, a_{\alpha_1\cdots\alpha_p} b_{\alpha_p\cdots\alpha_1}\ , \tag{2.3}$$

where the summation convention for the indices α has been used (note the reversal of the order of the indices in (2.3). Because of the skewsymmetry we could write

$$b_{\alpha_p\cdots\alpha_1} = (-)^\kappa\, b_{\alpha_1\cdots\alpha_p} \tag{2.4}$$

where $2\kappa = p$ or $p - 1$ according as p is even or odd).

The following three mappings of $\mathscr{V}(n)$ onto $\mathscr{V}(n)$ will be found useful:

(1) *Duality*. With every p form we can associate a q form, where $p + q = n$, by contraction with the unit n-form $\epsilon_{\alpha_1 \cdots \alpha_n}$ ($\epsilon_{123 \cdots n} = 1$). (The case $p = q = 2$ is the familiar case of the dual of an electromagnetic tensor [11]). We define

$$a^D_{\alpha_1 \cdots \alpha_p} = (i)^{\nu + 2\kappa} \epsilon_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} a_{\beta_1 \cdots \beta_q} / q!, \tag{2.5}$$

(where $2\nu = n$ or $n - 1$, according as n is even or odd; $2\kappa = 2\kappa_p = p$ or $p - 1$, according as p is even or odd). The numerical factor is chosen so that if a^D is the dual of a then a is the dual of a^D. This follows from

$$\begin{aligned} \epsilon_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} \epsilon_{\gamma_1 \cdots \gamma_p \beta_1 \cdots \beta_q} &= (-)^{pq} \epsilon_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} \cdot \epsilon_{\beta_1 \cdots \beta_q \gamma_1 \cdots \gamma_p} \\ &= p! \, \delta^{\alpha_1}_{[\gamma_1} \cdots \delta^{\alpha_p}_{\gamma_p]} . \end{aligned}$$

(Square brackets will be used to denote complete skewsymmetrisation, round brackets complete symmetrisation, as follows (e.g.):

$$A_{\alpha\beta\gamma} = \frac{1}{3!} (A_{\alpha\beta\gamma} + A_{\beta\gamma\alpha} + A_{\gamma\alpha\beta} - A_{\beta\alpha\gamma} - A_{\alpha\gamma\beta} - A_{\gamma\beta\alpha}),$$

$$A_{(\alpha\beta\gamma)} = \frac{1}{3!} (A_{\alpha\beta\gamma} + A_{\beta\gamma\alpha} + A_{\gamma\alpha\beta} + A_{\beta\alpha\gamma} + A_{\alpha\gamma\beta} + A_{\gamma\beta\alpha}).)$$

(2) A mapping P that multiplies all odd p-forms by -1,

$$\mathscr{A}^P = [(-)^p a_{\alpha_1 \alpha_p}]. \tag{2.7}$$

(3) A mapping K that reverses the order of indices in each p-form

$$\mathscr{A}^K = [(-)^\kappa a_{\alpha_1 \cdots \alpha_p}] = [a_{\alpha_p \cdots \alpha_1}]. \tag{2.8}$$

We observe the following important properties of all three mappings:

(a) They are *involutions*,

$$\mathscr{A}^{DD} = \mathscr{A}^{PP} = \mathscr{A}^{KK}, \tag{2.9}$$

(b) They commute with $SO(n, C)$ transformations of the n-fold tensor indices. The $SO(n, C)$ transformations also leave the scalar product invariant.

We list without proof some easily proved properties of the mappings D, P, K:

$$\mathscr{A}^D \cdot \mathscr{B} = \begin{cases} \mathscr{A} \cdot \mathscr{B}^D & \text{if } n \text{ is odd,} \\ \mathscr{A}^P \cdot \mathscr{B}^D & \text{if } n \text{ is even,} \end{cases} \tag{2.10}$$

$$\mathscr{A}^P \cdot \mathscr{B} = \mathscr{A} \cdot \mathscr{B}^P, \tag{2.11}$$

$$\mathscr{A}^K \cdot \mathscr{B} = \mathscr{A} \cdot \mathscr{B}^K, \tag{2.12}$$

$$\mathscr{A}^{KP} = \mathscr{A}^{PK}, \tag{2.13}$$

$$\mathscr{A}^{PD} = (-)^n \mathscr{A}^{DP}, \tag{2.14}$$

$$\mathscr{A}^{KD} = \begin{cases} (-)^\nu \mathscr{A}^{DK} & \text{if } n \text{ is odd,} \\ (-)^\nu \mathscr{A}^{PDK} & \text{if } n \text{ is even.} \end{cases} \tag{2.15}$$

Another important pair of linear mappings, which are not involutions, can be associated with *any unit vector* in the n-dimensional space ($\xi_\alpha \xi_\alpha = 1$). We define

(1) $$\mathscr{A}^Z = [a^Z_{\alpha_1 \cdots \alpha_p}],$$

where

$$a^Z_{\alpha_1 \cdots \alpha_p} = \xi_\alpha a_{\alpha \alpha_1 \cdots \alpha_p} . \tag{2.16}$$

The p form of $\mathscr{A}^Z$ is therefore constructed from the $p + 1$ form of $\mathscr{A}$. The n form of $\mathscr{A}^Z$ is identically zero.

(2) $$\mathscr{A}^{\bar{Z}} = [a^{\bar{Z}}_{\alpha_1 \cdots \alpha_p}],$$

where

$$a^{\bar{Z}}_{\alpha_1 \cdots \alpha_p} = p \xi_{[\alpha_1} a_{\alpha_2 \cdots \alpha_p]} . \tag{2.17}$$

The p form of $\mathscr{A}^{\bar{Z}}$ is constructed from the $p - 1$ form of $\mathscr{A}$ and the zero-form vanishes identically.

We obtain the following properties of Z and $\bar{Z}$ (which are most easily proved by taking $\xi_\alpha = \delta_\alpha{}^n$, which loses no generality because of the $SO(n, C)$ invariance of (2.16) and (2.17)).

(a) They are idempotent

$$\mathscr{A}^{ZZ} = \mathscr{A}^{\bar{Z}\bar{Z}} = 0. \tag{2.18}$$

(b) They satisfy

$$\mathscr{A}^{Z\bar{Z}} + \mathscr{A}^{\bar{Z}Z} = \mathscr{A} \tag{2.19}$$

(and the map $\mathscr{A}^{Z\bar{Z}} - \mathscr{A}^{Z\bar{Z}}$ corresponds in the n-space to a reflection in a hyperplane perpendicular to the unit vector ξ_α).

From Z and $\bar{Z}$ we can construct two more *involutions* as follows.

$$\mathscr{A}^X = \mathscr{A}^Z + \mathscr{A}^{\bar{Z}}, \tag{2.20}$$

$$\mathscr{A}^Y = \frac{1}{i}(\mathscr{A}^Z - \mathscr{A}^{\bar{Z}}) \tag{2.21}$$

then we find

$$\mathscr{A}^{XX} = \mathscr{A}^{YY} = \mathscr{A}. \tag{2.22}$$

We can now derive relations connecting the involutions X and Y with D, P and K and the scalar product. For example,

$$\begin{gathered} \mathscr{A} \cdot \mathscr{B}^Z = \mathscr{A}^{\bar{Z}} \cdot \mathscr{B}, \\ \mathscr{A}^{KZ} = \mathscr{A}^{ZKP}, \\ \mathscr{A}^{K\bar{Z}} = -\mathscr{A}^{\bar{Z}KP}, \\ \mathscr{A}^{PZ} = \mathscr{A}^{ZP}, \qquad \mathscr{A}^{P\bar{Z}} = -\mathscr{A}^{\bar{Z}P}, \\ \mathscr{A}^{DZ} = \mathscr{A}^{\bar{Z}D}, \end{gathered} \tag{2.23}$$

from which we can obtain

$$\mathscr{A}^X \cdot \mathscr{B} = \mathscr{A} \cdot \mathscr{B}^X, \qquad \mathscr{A}^Y \cdot \mathscr{B} = -\mathscr{A} \cdot \mathscr{B}^Y, \tag{2.24}$$

$$\mathscr{A}^{PX} = \mathscr{A}^{XP}, \qquad \mathscr{A}^{PY} = -\mathscr{A}^{YP}, \tag{2.25}$$

$$\mathscr{A}^{KX} = i\mathscr{A}^{YKP}, \qquad \mathscr{A}^{KY} = -i\mathscr{A}^{XKP}, \tag{2.26}$$

$$\mathscr{A}^{DX} = \mathscr{A}^{XD}, \qquad \mathscr{A}^{DY} = -\mathscr{A}^{YD}. \tag{2.28}$$

3. Even Clifford Algebras

(The reader is referred to the literature for detailed discussions and proofs of our assertions about the general properties of the Clifford algebras [2–5, 9].)

The basis of the Clifford algebra $C_{2\nu}$ is the set of 4^ν linearly independent products of 2ν generators G_α (*the generators*) which anticommute with each other and have unit squares,

$$G_{(\alpha}G_{\beta)} = \delta_{\alpha\beta} \qquad (\alpha, \beta = 1,\dots, 2\nu). \tag{3.1}$$

Define the set of skewsymmetrised products

$$\begin{aligned} &1,\ G_\alpha,\ G_{\alpha\beta} = G_{[\alpha}G_{\beta]}, \\ &\qquad G_{\alpha\beta\gamma} = G_{[\alpha\beta}G_{\gamma]}, \qquad (\alpha, \beta \cdots = 1,\dots, 2\nu). \\ &\qquad \cdots \\ &G_{\alpha_1\cdots\alpha_2} \end{aligned} \tag{3.2}$$

Then the

$$\binom{2\nu}{0} + \binom{2\nu}{1} + \cdots + \binom{2\nu}{2\nu} = 4^\nu$$

quantities (3.2) are linearly independent (provided no other restriction than (3.1) is imposed on the generators), and taking them as the basis of the algebra, a general element can be written

$$A = a + a_\alpha G_\alpha + \frac{1}{2} a_{\alpha\beta} G_{\alpha\beta} + \cdots + \frac{1}{(2\nu)!} a_{\alpha_1 \cdots \alpha_2} G_{\alpha_1 \cdots \alpha_2}. \tag{3.3}$$

Using the notation of Section 1, we may write

$$\mathscr{G} = [G_{\alpha_1 \cdots \alpha_p}]$$

for the set of base elements just as though they were a vector of $\mathscr{V}(2\nu)$, though the components in this case are not complex numbers, but operators. The components of A with respect to this basis form a vector $\mathscr{A}$ of $\mathscr{V}(2\nu)$ and we can rewrite (3.3) in the symbolic form

$$A = \mathscr{A} \cdot \mathscr{G}^K. \tag{3.4}$$

The *even* Clifford algebra $C_{2\nu}$ has only one irreducible representation, which is 2^ν dimensional. We shall assume throughout that the G_α are expressed by matrices of this representation. It is a faithful representation so we lose no generality through this assumption. Then the element A is a $2^\nu \times 2^\nu$ matrix, and since a set of 4^ν linearly independent matrices span all $2^\nu \times 2^\nu$ matrices (with complex coefficients), *any* complex $2^\nu \times 2^\nu$ matrix can be written in the form (3.3) ((3.4)). The representation will also be supposed to be in a form in which the G_α are all Hermitian (an explicit method of constructing a set of $2^\nu \times 2^\nu$ Hermitian generators will be given in Section 6).

We now derive an expression for the product of any pair of base elements as a linear combination of base elements (i.e., we shall obtain the structure constants of the algebra). The required expression is

$$G_{\alpha_k \cdots \alpha_2 \alpha_1} G_{\beta_1 \beta_2 \cdots \beta_l} = \sum_{p=0}^{\min(k,l)} p! \binom{k}{p} \binom{l}{p} S \delta_{\alpha_1 \beta_1} \ldots \delta_{\alpha_p \beta_p} \ldots \cdot G_{\alpha_k \cdots \alpha_{p+1} \beta_{p+1} \cdots \beta_l}, \tag{3.5}$$

where S skewsymmetrises the α's and β's separately, in the qualities occurring to the right,

$$(SA_{\alpha_1 \cdots \alpha_k \beta_1 \cdots \beta} = A_{[\alpha_1 \cdots \alpha_k][\beta_1 \cdots \beta]}).$$

Of course, any product of generators occurring on the right with more than 2ν indices is identically zero.

Proof of (3.5).

$$k!l!S\,\delta_{\alpha_1\beta_1}\ldots\delta_{\alpha_p\beta_p}G_{\alpha_k\cdots\alpha_{p+1}\beta_{p+1}\cdots\beta_l}=\delta_{\alpha_1\beta_1}\ldots\delta_{\alpha_p\beta_p}G_{\alpha_k\cdots\alpha_{p+1}\beta_{p+1}\cdots\beta_l}+\cdots, \tag{3.6}$$

where the right-hand side has $k!l!$ terms obtained from the first by all the permutations of the α's and of the β's. There are $p!$ ways of permuting the δ symbols in the product $\delta_{\alpha_1\beta_1}\ldots\delta_{\alpha_p\beta_p}$ without altering it, $(k-p)!$ ways of permuting the α's in $G_{\alpha_k\cdots\alpha_{p+1}\beta_{p+1}\beta}$ and $(l-p)!$ ways of permuting the β's. Hence $p!(k-p)!\,(l-p)!$ of the terms on the right of (3.6) are equal to the term written explicitly. Now suppose the indices satisfy the following condition (C): $\alpha_i=\beta_i$ $(i=1,\ldots p)$, the remaining α's and β's all different from these and from each other. In this case, we have

$$k!l!S\,\delta_{\alpha_1\beta_1}\ldots\delta_{\alpha_p\beta_p}G_{\alpha_k\cdots\alpha_{p+1}\beta_{p+1}\cdots\beta_l}=p!(k-p)!\,(l-p)!\,G_{\alpha_k\cdots\alpha_{p+1}\alpha_{p+1}\cdots\alpha_l}\,.$$

But also for the index sets satisfying (C),

$$G_{\alpha_k\cdots\alpha_1}G_{\beta_1\cdots\beta_l}=G_{\alpha_k\cdots\alpha_{p+1}\beta_{p+1}\cdots\beta_l}\,.$$

Hence, under condition (C),

$$G_{\alpha_k\cdots\alpha_1}G_{\beta_1\cdots\beta_l}=p!\binom{k}{p}\binom{l}{p}\,S\delta_{\alpha_1\beta_1}\ldots\delta_{\alpha_p\beta_1}G_{\alpha_k\cdots\alpha_{p+1}\beta_{p+1}\cdots\beta_l}\,.$$

Now because of the skewsymmetry in α's and β's on both sides, the above expression, if valid for (C) must be valid for all index sets for which any p of the α's are equal to any p of the α's, and the rest all different from these and from each other.

Moreover, the right hand side vanishes identically if the condition is not satisfied, so that the condition can be dispensed with by replacing the right hand side by a *sum* of similar expressions for all possible values of p. This completes the proof of (3.5).

To clarify the significance of (3.5), which is rather cumbersome in its general form, we will take the special case $k=l=2$, which gives

$$G_{\alpha\beta}G_{\gamma\delta}=G_{\alpha\beta\gamma\delta}+(\delta_{\beta\gamma}G_{\alpha\delta}-\delta_{\alpha\gamma}G_{\beta\delta}+\delta_{\alpha\delta}G_{\beta\gamma}-\delta_{\beta\delta}G_{\alpha\gamma})+(\delta_{\beta\gamma}\delta_{\alpha\delta}-\delta_{\alpha\gamma}\delta_{\beta\delta})$$

from which we obtain

$$[G_{\alpha\beta}\,,\,G_{\gamma\delta}]=2(\delta_{\beta\gamma}G_{\alpha\delta}-\delta_{\alpha\gamma}G_{\beta\delta}+\delta_{\alpha\delta}G_{\beta\gamma}-\delta_{\beta\delta}G_{\alpha\gamma}), \tag{3.7}$$

which is the well-known result that $\frac{1}{2}\,G_{\alpha\beta}$ are infinitesimal generators of a 2^ν-dimensional representation of $SO(2\nu,c)$. Another useful formula is obtained by taking the trace of (3.5). Since the only base element with nonzero trace is the unit element, we get

$$\frac{1}{2^\nu} \operatorname{Tr} G_{\alpha_k \cdots \alpha_1} G_{\beta_1 \cdots \beta_l} = k!\, \delta_{kl} \delta^{\beta_1}_{[\alpha_1} \cdots \delta^{\beta_k}_{\alpha_k]} , \tag{3.8}$$

and, therefore,

$$\frac{1}{2^\nu} \operatorname{Tr} A G_{\beta_1 \cdots \beta_p} = a_{\beta_p \cdots \beta_1} = (-)^\kappa\, a_{\beta_1 \cdots \beta_p} , \tag{3.9}$$

and

$$\frac{1}{2^\nu} \operatorname{Tr} AB = \sum_{p=0}^{2\nu} \frac{1}{p!}\, a_{\alpha_1 \cdots \alpha_p} b_{\alpha_p \cdots \alpha_1} \tag{3.10}$$

for any two elements A, B of the Clifford algebra. This gives a formulation of the scalar product of $\mathscr{V}(2\nu)$ in terms of the operations of $C_{2\nu}$:

$$\mathscr{A} \cdot \mathscr{B} = \frac{1}{2^\nu} \operatorname{Tr} AB. \tag{3.11}$$

Of special importance is the matrix G formed from the product all the generators

$$G = (-\,i)^\nu\, G_1 G_2 \cdots G_{2\nu} . \tag{3.12}$$

The numerical factor being chosen so that $G^2 = 1$ and G is Hermitian if the generators are. G is the "dual" of $G_{\alpha_1 \cdots \alpha_{2\nu}}$ apart from a numerical factor

$$G = (-\,i)^\nu\, \epsilon_{\alpha_1 \cdots \alpha_2} G_{\alpha_1 \cdots \alpha_2} / (2\nu)! \tag{3.13}$$

From this expression we obtain

$$\begin{aligned}
G_{\alpha_1} G &= (-\,i)^\nu\, \epsilon_{\alpha_1 \cdots \alpha_{2\nu}} G_{\alpha_2 \cdots \alpha_{2\nu}} / (2\nu - 1)! \\
G_{\alpha_1 \alpha_2} G &= -\,(-\,i)^\nu\, \epsilon_{\alpha_1 \cdots \alpha_{2\nu}} G_{\alpha_3 \cdots \alpha_{2\nu}} / (2\nu - 2)! \\
&\cdots \\
G_{\alpha_1 \cdots \alpha_2} G &= (i)^\nu\, \epsilon_{\alpha_1 \cdots \alpha_{2\nu}} .
\end{aligned}$$

The general form is

$$G_{\alpha_1 \cdots \alpha_p} G = (-)^p\, G G_{\alpha_1 \cdots \alpha_p} = (-)^\nu\, (i)^{\nu + 2\kappa}\, \epsilon_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} G_{\beta_1 \cdots \beta_q} / q! \tag{3.14}$$

In other words, multiplication of the set of base elements by G takes the *dual*, in the following sense:

$$\mathscr{G} G = G \mathscr{G}^P = (-)^\nu\, \mathscr{G}^D. \tag{3.15}$$

Thus, for the general element (3.4), using the relations (2.10–15),

$$\begin{aligned} AG &= \mathscr{A}^K \cdot \mathscr{G}G = (-)^\nu \mathscr{A}^K \cdot \mathscr{G}^D = (-)^\nu \mathscr{A}^{KD} \cdot \mathscr{G}^P \\ &= \mathscr{A}^{PDK} \cdot \mathscr{G}^P = \mathscr{A}^{DKP} \cdot \mathscr{G}^P = \mathscr{A}^D \cdot \mathscr{G}^K, \end{aligned} \tag{3.16}$$

and

$$GA = \mathscr{A}^{PD} \cdot \mathscr{G}^K. \tag{3.17}$$

Thus, if the element A has components $\mathscr{A}$, then AG has components $\mathscr{A}^D$ (and GA has components $\mathscr{A}^{PD} = \mathscr{A}^{DP}$, so that GAG has components $\mathscr{A}^P$).

We may therefore say that multiplication by G in the Clifford algebra corresponds to the duality mapping in $\mathscr{V}(2\nu)$. It is clear that multiplication by any X satisfying $X^2 = 1$ in C_2 corresponds to an involuntary mapping in $\mathscr{V}(2\nu)$. For example, let $X = \xi_\alpha G_\alpha$, where $\xi_\alpha \xi_\alpha = 1$. Then

$$XG_{\beta_1 \cdots \beta_p} = \xi_\alpha G_{\alpha\beta_1 \cdots \beta_p} + p\xi_{[\beta_1} G_{\beta_2 \cdots \beta_p]} \,. \tag{3.18}$$

In the notation of Section 2, this is just

$$X\mathscr{G} = \mathscr{G}^X. \tag{3.19}$$

If a general element A of the algebra has components $\mathscr{A}$, then XA has components $\mathscr{A}^X$. Finally, for the sake of completeness we state that the analogous results for multiplication on the *right* by the matrix $X = \xi_\alpha G_\alpha$:

$$\mathscr{G}X = i\mathscr{G}^{YP} \tag{3.20}$$

and AX has components $i\mathscr{A}^{PY}$. Because the involutions X and Y on $\mathscr{V}(2\nu)$ are thus represented by multiplications on the *left* and on the *right*, respectively, in C_2, we easily obtain, by evaluating XAX, the relation

$$\mathscr{A}^{KY} = -\mathscr{A}^{YK}. \tag{3.21}$$

An important property of the matrix G is that it anticommutes with all the base elements, and this is the reason why we lose no generality by considering only the *even* Clifford algebras. Writing $N = 2\nu + 1$ and defining $G_N = G$ we can generalize (3.1) to

$$G_{(\alpha} G_{\beta)} = \delta_{\alpha\beta} \qquad (\alpha, \beta = 1 \cdots N) \tag{3.22}$$

so that the 2^ν-dimensional representation of $C_{2\nu}$ already contains an irreducible representation of the Clifford algebra C_N; we simply identify G as the extra generator. Of course, the representation is not faithful for C_N. There are *two* inequivalent irreducible representations of an odd Clifford algebra, the other one being obtained by taking $-G$ as the extra generator. The 2.2^ν-dimensional representation of C_N formed from the direct sum of the two irreducible representations is faithful.

4. N-Dimensional Tensors and $C_{2\nu}$

We now formally enlarge the set of base elements (3.2) by including $G_N = G(N = 2\nu + 1)$ in the construction

$$1, G_\alpha, G_{\alpha\beta}, ..., G_{\alpha_1 \cdots \alpha_N} \qquad (\alpha, \beta = 1, ..., N). \tag{4.1}$$

Of course, these quantities are not linearly independent. Nevertheless, (4.1) will be useful for giving an alternative formulation of the properties of $C_{2\nu}$ in terms of $\mathscr{V}(N)$ rather than $\mathscr{V}(2\nu)$. We write the general element in the form

$$A = \frac{1}{2}\left(a + a_\alpha G_\alpha + \cdots + \frac{1}{N!} a_{\alpha_1 \cdots \alpha_N} G_{\alpha_1 \cdots \alpha_N}\right). \tag{4.2}$$

The linear dependence of the set (4.1) will manifest itself in a set of restriction on the set of components

$$\mathscr{A}' = [a_{\alpha_1 \cdots \alpha_p}] \tag{4.3}$$

in $\mathscr{V}(N)$. Writing the set (4.1) as $\mathscr{G}'$ we have the following shorthand form for (4.2) in the notation of $\mathscr{V}(N)$

$$A = \tfrac{1}{2} \mathscr{A}' \cdot \mathscr{G}'^K. \tag{4.4}$$

The expression (3.14) can be rewritten using the N-fold indices α; in this form it expresses the *linear dependence* of (4.1):

$$G_{\alpha_1 \cdots \alpha_p} = (-)^\nu (i)^{2\kappa+\nu} \epsilon_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} G_{\beta_1 \cdots \beta_q}/q! \qquad (\alpha, \beta = 1, ..., N). \tag{4.5}$$

That is,

$$\mathscr{G}' = (-)^\nu \mathscr{G}'^D. \tag{4.6}$$

We therefore have

$$A = \tfrac{1}{2} \mathscr{A}'^K \cdot \mathscr{G}' = \tfrac{1}{2} (-)^\nu \mathscr{A}'^K \cdot \mathscr{G}'^D = \tfrac{1}{2} \mathscr{A}'^D \cdot \mathscr{G}'^K, \tag{4.7}$$

so that the set of components $\mathscr{A}'$ of a general element, considered as a vector of $\mathscr{V}(N)$, can be taken to satisfy the *self-duality* condition

$$\mathscr{A} = \mathscr{A}'^D. \tag{4.8}$$

Because of the self-duality, only those p-forms up to rank ν need be specified. The rest are given in terms of them by (4.8). Thus (5.2) can be rewritten as

$$A = a + a_\alpha G_\alpha + \cdots + \frac{1}{\nu!} a_{\alpha_1 \cdots \alpha_\nu} G_{\alpha_1 \cdots \alpha_\nu} \qquad (\alpha = 1, ..., N). \tag{4.9}$$

The total number of components in this expression being

$$\binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{\nu} = \frac{1}{2}\, 2^N = 4^\nu$$

as required.

Alternatively, we need specify only these components of the p-forms ($p = 0,\ldots, N$) which do not have the value N among their indices, the remaining components can be obtained from these by (4.8). The expression (4.2) then takes on the form

$$A = a + a_\alpha G_\alpha + \cdots + \frac{1}{(2\nu)!}\, a_{\alpha_1\cdots\alpha_{2\nu}} G_{\alpha_1\cdots\alpha_{2\nu}} \qquad (\alpha = 1,\ldots, 2\nu), \tag{4.10}$$

so that we have recovered the original formulation in terms of $\mathscr{V}(2\nu)$. The factor $\frac{1}{2}$ in (4.2) was introduced so that this equivalence would hold.

Finally, we note that (3.5) is valid also when the indices $\alpha, \beta \cdots$ are taken to be the N-fold indices ((3.5) is a consequence of (3.1) and (3.2) which still hold good for the quantities (4.1)). Hence we have the result that $\frac{1}{2}\, G_{\alpha\beta}$ ($\alpha, \beta = 1,\ldots, N$) are generators of a representation of $SO(N, C)$. However, when we take the trace of (3.5) we have to take account of the fact that

$$G_{\alpha_1\cdots\alpha_N} = (i)^\nu\, \epsilon_{\alpha_1\cdots\alpha_N} \cdot 1$$

has non-zero trace. In place of (3.8) we get, for N-fold indices,

$$\frac{1}{2^\nu}\, \mathrm{Tr}\, G_{\alpha_k\cdots\alpha_1} G_{\beta_1\cdots\beta_l} = \delta_{kl}(k!\, \delta^{\beta_1\ldots\beta_k}_{[\alpha_1\cdots\alpha_k]}) + (i)^\nu\, \delta_{k+l,N}\epsilon_{\alpha_k\cdots\alpha_1\beta_1\cdots\beta_l}\,, \tag{4.11}$$

and, therefore, for $A = \frac{1}{2}\mathscr{A}' \cdot \mathscr{G}'^K$, $B = \frac{1}{2}\mathscr{B}' \cdot \mathscr{G}'^K$,

$$\begin{aligned} \frac{1}{2^\nu}\, \mathrm{Tr}\, A G_{\beta_p\cdots\beta_1} &= \frac{1}{2}\,(a_{\beta_1\cdots\beta_p} + a^D_{\beta_1\cdots\beta_p}) \\ &= a_{\beta_1\cdots\beta_p}\,, \end{aligned} \tag{4.12}$$

and

$$\frac{1}{2^\nu}\, \mathrm{Tr}\, AB = \frac{1}{2}\, \mathscr{A}' \cdot \mathscr{B}'. \tag{4.13}$$

5. $(N+1)$-Dimensional Tensors

We now describe a further formulation of $C_{2\nu}$ in terms of the vector space $\mathscr{V}(M)$, where $M = N + 1 = 2\nu + 2$. For this purpose we introduce the symbol

$$G_M = -\, i \tag{5.1}$$

and use Latin indices $a, b\ldots$ which run from 1 to M (the Greek indices being reserved for the range 1 to N as in the previous section):

$$G_a = (G_\alpha\,,\, -i). \tag{5.2}$$

We also define

$$\bar{G}_a = (G_\alpha\,, i) \tag{5.3}$$

(which is a generalization of the concept of quaternion conjugation (the case $\nu = 1$, $M = 4$)). For Hermitian generators, $\bar{G}_a = G_a{}^+$.

Because G_M *commutes* with all the generators and has square -1 we can generalize (3.18) still further in two ways:

$$\begin{aligned} G_{(a}\bar{G}_{b)} &= \delta_{ab}\,, \\ \bar{G}_{(a}G_{b)} &= \delta_{ab}\,, \end{aligned} \tag{5.4}$$

and we can construct skewsymmetrised products

$$\begin{aligned} 1, G_a\,, G_{ab} &= G_{[a}\bar{G}_{b]}\,, \\ G_{abc} &= G_{[ab}G_{c]}\,, \\ G_{abcd} &= G_{[abc}\bar{G}_{d]}\,, \\ &\cdots \\ G_{a_1\cdots a_M}&\,, \end{aligned} \tag{5.5}$$

or, alternatively,

$$\begin{aligned} 1, \bar{G}_a\,, \bar{G}_{ab} &= \bar{G}_{[a}G_{b]}\,, \\ \bar{G}_{abc} &= \bar{G}_{[ab}\bar{G}_{c]}\,, \\ \bar{G}_{abcd} &= \bar{G}_{[abc}G_{d]}\,, \\ &\cdots \\ \bar{G}_{a_1\cdots a_M}&\,. \end{aligned} \tag{5.6}$$

(The rule operating here is that a product of "generators" G_a is formed with a bar over *alternate* matrices. The reason will become apparent when $SO(M, C)$ transformation laws are assigned to these quantities). A partial expression of the linear dependence of the quantities (5.5) is obtained by manipulating (4.5) to get

$$G_{a_1\cdots a_p} = (-)^p\,(-)^\kappa\,(-i)^\mu\,\epsilon_{a_1\cdots a_p b_1\cdots b_q} G_{b_1\cdots b_q}/q!, \tag{5.7}$$

where $q = M - p$ and $\mu = \nu + 1 = M/2$. Writing the set (5.5) as an element $\mathscr{G}$ of $\mathscr{V}(M)$ (not to be confused with the $\mathscr{G}$ of $\mathscr{V}(2\nu)$ in Section 3), the above expression is just

$$\mathscr{G} = (-)^\mu\,\mathscr{G}^{PD} = (-)^\mu\,\mathscr{G}^{DP}. \tag{5.8}$$

A further linear dependence is given by

$$\begin{aligned} G_M &= -i, \\ G_{M\alpha} &= -iG_\alpha, \\ G_{M\alpha\beta} &= -iG_{\alpha\beta}, \text{ etc.,} \end{aligned} \tag{5.9}$$

which can be written as a single relation

$$G_{a_1\cdots a_p} = i(\xi_a G_{aa_1\cdots a_p} - p\xi_{[a_1}G_{a_2\cdots a_p]}), \tag{5.10}$$

where ξ_a is the unit vector in the M direction, $\xi_a = \delta_{aM}$, which, in the notation of $\mathscr{V}(M)$, is just

$$\mathscr{G} = -\mathscr{G}^Y. \tag{5.11}$$

The linear dependence of the set $\bar{\mathscr{G}}$, (5.6), is expressed by relations analogous to (5.8) and (5.11):

$$\bar{\mathscr{G}} = -(-)^\mu \bar{\mathscr{G}}^{PD} = -(-)^\mu \bar{\mathscr{G}}^{DP} \tag{5.12}$$

and

$$\bar{\mathscr{G}} = \bar{\mathscr{G}}^Y. \tag{5.13}$$

We now use the formalism of Section 2 to re-express these linear dependence relations as restrictions on the $\mathscr{V}(M)$ components of a general element

$$A = \tfrac{1}{4}\mathscr{A}\cdot\mathscr{G}^K = \tfrac{1}{4}\bar{\mathscr{A}}\cdot\bar{\mathscr{G}}^K. \tag{5.14}$$

We find that $\mathscr{A}$ is *self-dual:*

$$\mathscr{A} = \mathscr{A}^D, \tag{5.15}$$

and that

$$\mathscr{A} = i\mathscr{A}^{PX}. \tag{5.16}$$

Also $\bar{\mathscr{A}} = -\bar{\mathscr{A}}^D = -i\bar{\mathscr{A}}^{PX}$. We now define four alternative *linearly independent* sets of base elements as follows:

(a) A set $\mathscr{G}^{(1)}$ whose odd p forms and p forms with $p > \mu = M/2$ are zero:

(i) μ even

$$\mathscr{G}^{(1)} = [1, G_{ab}, G_{abcd}, \ldots, \tfrac{1}{2}G_{a_1\cdots a_\mu}] \tag{5.17}$$

with

$$G_{a_1\cdots a_\mu} = G^D_{a_1\cdots a_\mu}.$$

(ii) μ odd

$$\mathscr{G}^{(1)} = [1, G_{ab}, G_{abcd}, \ldots, G_{a_1 \cdots a_\nu}]. \tag{5.18}$$

(b) A set $\mathscr{G}^{(2)}$ whose even p forms and p forms with $p > M/2$ are zero.

(i) μ even

$$\mathscr{G}^{(2)} = [G_a, G_{abc}, \ldots, G_{a_1 \cdots a_\nu}] \tag{5.19}$$

(ii) μ odd

$$\mathscr{G}^{(2)} = [G_a, G_{abc}, \ldots, \tfrac{1}{2} G_{a_1 \cdots a_\mu}]$$

with

$$G_{a_1 \cdots a_\mu} = G^D_{a_1 \cdots a_\mu}. \tag{5.20}$$

(c) A set $\bar{\mathscr{G}}^{(1)}$ constructed like $\mathscr{G}^{(1)}$ but with the "barred" quantities. For even μ the condition on $G_{a_1 \cdots a_\mu}$ is replaced by

$$\bar{G}_{a_1 \cdots a} = - \bar{G}^D_{a_1 \cdots a_\mu} \tag{5.21}$$

(d) A set $\bar{\mathscr{G}}^{(2)}$ constructed like $\mathscr{G}^{(2)}$. For μ odd we have

$$\bar{G}_{a_1 \cdots a_\mu} = - \bar{G}^D_{a_1 \cdots a_\mu}. \tag{5.22}$$

(That each of these sets are linearly independent is shown by the following expressions for the number of quantities

$$\begin{aligned} &\binom{M}{0} + \binom{M}{2} + \binom{M}{4} + \cdots + \binom{M}{\mu - 2} + \tfrac{1}{2}\binom{M}{\mu} \\ &= \binom{M}{1} + \binom{M}{3} + \binom{M}{5} + \cdots + \binom{M}{\mu - 1} \\ &= 4^\nu \qquad \text{for } \mu = M/2 = \nu + 1 \text{ even,} \end{aligned}$$

and

$$\begin{aligned} &\binom{M}{0} + \binom{M}{2} + \cdots + \binom{M}{\mu - 1} \\ &= \binom{M}{1} + \binom{M}{3} + \cdots + \binom{M}{\mu - 2} + \tfrac{1}{2}\binom{M}{\mu} \\ &= 4^\nu \qquad \text{for } \mu = M/2 = \nu + 1 \text{ odd,} \end{aligned}$$

and by the fact that $\mathscr{G}^{(1)}$, $\mathscr{G}^{(2)}$, $\bar{\mathscr{G}}^{(1)}$ or $\bar{\mathscr{G}}^{(2)}$ span all $2^\nu \times 2^\nu$ matrices).

From the linear dependence relations on $\mathscr{G}$ and on $\bar{\mathscr{G}}$ we can obtain the following properties

$$\begin{gathered}\mathscr{G}^{(2)} = -\mathscr{G}^{(1)Y}, \qquad \mathscr{G}^{(1)} = -\mathscr{G}^{(2)Y}, \\ \mathscr{G} = (\mathscr{G}^{(1)} + \mathscr{G}^{(2)}) + (-)^{\mu}(\mathscr{G}^{(1)} - \mathscr{G}^{(2)})^{D},\end{gathered} \tag{5.23}$$

and similarly

$$\begin{gathered}\bar{\mathscr{G}}^{(2)} = \bar{\mathscr{G}}^{(1)Y}, \qquad \bar{\mathscr{G}}^{(1)} = \bar{\mathscr{G}}^{(2)Y}, \\ \bar{\mathscr{G}} = (\bar{\mathscr{G}}^{(1)} + \bar{\mathscr{G}}^{(2)}) - (-)^{\mu}(\bar{\mathscr{G}}^{(1)} - \bar{\mathscr{G}}^{(2)})^{D},\end{gathered} \tag{5.24}$$

from which we can obtain, for a general element,

$$A = \tfrac{1}{4}\mathscr{A}^{K} \cdot \mathscr{G} = \mathscr{A}^{K} \cdot \mathscr{G}^{(1)} = \mathscr{A}^{K} \cdot \mathscr{G}^{(2)}, \tag{5.25}$$

$$A = \tfrac{1}{4}\bar{\mathscr{A}}^{K} \cdot \bar{\mathscr{G}} = \bar{\mathscr{A}}^{K} \cdot \bar{\mathscr{G}}^{(1)} = \bar{\mathscr{A}}^{K} \cdot \bar{\mathscr{G}}^{(2)}, \tag{5.26}$$

which proves our previous assertion that any one of the four sets $\mathscr{G}^{(1)}$, $\mathscr{G}^{(2)}$, $\bar{\mathscr{G}}^{(1)}$, $\bar{\mathscr{G}}^{(2)}$ span all $2^{\nu} \times 2^{\nu}$ matrices.

For convenience we shall call a general element A of the Clifford algebra $C_{2\nu}$ a *quantity* of *kind* 1, 2, 3, or 4 according to whether it is expressed in terms of the basis $\mathscr{G}^{(1)}$, $\mathscr{G}^{(2)}$, $\bar{\mathscr{G}}^{(1)}$ or $\bar{\mathscr{G}}^{(2)}$, respectively. Retaining the rule regarding products introduced in the paragraph following (5.6) we obtain the following multiplication Table I for the multiplication of a quantity of kind i by one of kind j to give a quantity of kind k. "Forbidden" products are indicated by—.

TABLE I

	1	2	3	4
1	1	2	—	—
2	—	—	2	1
3	—	—	3	4
4	4	3	—	—

If we use two kinds of capital Latin indices for the row and column labels of of a matrix A the above table can be reproduced by the following assignments, provided we introduce the rule that a pair of capital Latin indices can be contracted only if they are of the same type, one up and one down:

$$\begin{array}{rl} \text{For kind 1:} & A_{A}{}^{B} \\ 2: & A_{A\bar{B}} \\ 3: & A_{\bar{B}}{}^{\bar{A}} \\ 4: & A^{\bar{A}B} \end{array} \tag{5.27}$$

(Then the index types of G_a and $\bar{G}_a$ are $G_{aA\bar{B}}$ and $\bar{G}_a^{\bar{A}B}$ and everything follows from this.) Our aim is to relate the different kinds of capital Latin indices (which we will call "spinor" indices) to $SO(M, C)$ transformation laws.

With the above rules in mind, the multiplication law (3.5) is generalizable to the M fold indices

$$G_{a_k \cdots a_1} G_{b_1 \cdots b_l} = \sum_{p=0}^{\min(k,l)} \binom{k}{p} \binom{l}{p} p! \; S[\delta_{a_1 b_1} \cdots \delta_{a_p b_p} G_{a_k \cdots a_{k+1} b_{k+1} \cdots b_l}] \tag{5.28}$$

provided bars are written over the G's wherever necessary to satisfy the above rules. For $k = l = 2$ we obtain the result that

$$\tfrac{1}{2} G_{ab} \qquad \text{and} \qquad \tfrac{1}{2} \bar{G}_{ab} \tag{5.29}$$

are infinitesimal generators of two 2^ν-dimensional representations of $SO(M, C)$. Their irreducibility follows from the irreducibility of $\frac{1}{2} G_{\alpha\beta}$ ($\alpha, \beta = 1,\ldots, 2\nu$) and their inequivalence follows from the nonexistence of a matrix M satisfying $MG_a M^{-1} = - \bar{G}_a$. The G_{ab} (and $\bar{G}_{ab}$) are all traceless so that the spin transformations of $SO(M, C)$ belong to $SL(2^\nu, C)$. The $G_{ab}(\bar{G}_{ab})$ are also skewhermitian, so that the subgroup with real parameters is unitary: the transformations of the 2^ν-component spinor representations of the *real* orthogonal group $SO(M)$ belong to $SU(2^\nu)$. The representations (5.29) are faithful representations except when $\nu = 0$ or 1. For $\nu = 1$ we have an interesting special case, the two component spinor representations of $SO(4, C)$. From (5.17) and (5.21) we see that G_{ab} is self-dual, $\bar{G}_{ab}$ antiself-dual: $G_{ab} = \epsilon_{abcd} G_{cd}$, $\bar{G}_{ab} = - \epsilon_{abcd} \bar{G}_{cd}$.

6. Construction of Hermitian Representations

From a given 2^ν-dimensional irreducible Hermitian representation of $C_{2\nu}$ it is possible to construct a 2^μ-dimensional irreducible Hermitian representation of $C_{2\mu}$ ($\mu = \nu + 1$). Writing $M = 2\nu + 2$ we have the 2^ν-dimensional matrices

$$\begin{aligned} &G_a \qquad (a = 1,\ldots, M), \\ &\bar{G}_a \qquad (a = 1,\ldots, M), \end{aligned} \tag{6.1}$$

associated with the 2^ν-dimensional irreducible representation of $C_{2\nu}$ in the manner previously discussed. As is easily verified, the matrices

$$F_a = \begin{pmatrix} & G_a \\ \bar{G}_a & \end{pmatrix} \qquad (a = 1,\ldots, M) \tag{6.2}$$

are Hermitian and satisfy

$$F_{(a}F_{b)} = \delta_{ab} \qquad (a, b = 1, \dots, M), \tag{6.3}$$

and so they are the generators of the required 2^{μ}-dimensional representation of $C_{2\mu} = C_M$. We can then construct

$$\begin{aligned} F_{M+1} &= (-i)^{\mu} F_1 F_2 \cdots F_M , \\ F_{M+2} &= -i, \end{aligned} \tag{6.4}$$

and we then have the quantities

$$F_a , \bar{F}_a \qquad (a = 1, \dots, M+2) \tag{6.5}$$

for C_M analogous to the quantities (6.1) for $C_{2\nu}$. It is interesting to note that

$$\begin{aligned} F_{M+1} &= (-i)^{\mu} F_1 \cdots F_M = (-i)(-i)^{\nu} F_1 \cdots F_{2\nu} F_N F_M \\ &= (-i)(-i)^{\nu} \begin{pmatrix} G_1 \cdots G_{2\nu} & \\ & G_1 \cdots G_{2\nu} \end{pmatrix} \begin{pmatrix} & G_N \\ G_N & \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} \\ &= (-i) \begin{pmatrix} G_N & \\ & G_N \end{pmatrix} \begin{pmatrix} & G_N \\ G_N & \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} \\ &= \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \end{aligned} \tag{6.6}$$

Thus, if a hierarchy of representations of the even Clifford algebras is constructed by the above method, the matrix G defined by (3.12) for any $C_{2\nu}$ has the simple diagonal form $\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$ (where 1 represents the $2^{\nu}/2$-dimensional unit matrix).

It is instructive to carry out this process for the first few steps, starting from the trivial algebra C_0 which consists of the 1-dimensional matrices

$$G_1 = G_N = 1, \qquad G_2 = G_M = -i. \tag{6.7}$$

Then, for C_2 we get the *Pauli matrices*

$$F_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad F_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad F_4 = -i, \tag{6.8}$$

writing

$$F_a = (\boldsymbol{\sigma}, -i) = \sigma_a \qquad \text{and} \qquad \bar{F}_a = (\boldsymbol{\sigma}, i) = -\bar{\sigma}_a , \tag{6.9}$$

the next stage in the building-up process gives us the following representation of C_4.

$$F_a = \begin{pmatrix} & \sigma_a \\ -\bar{\sigma}_a & \end{pmatrix} \qquad (a = 1, ..., 4),$$
$$F_5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad F_6 = -i. \tag{6.10}$$

The matrices $\gamma_a = F_a F_5$ satisfying $\gamma_{(a}\gamma_{b)} = -\delta_{ab}$ are seen to be just the Dirac matrices in the Weyl representation [11] and $\gamma_5 = F_5$.

The hierarchy of Clifford algebra representations constructed here is the same as that given by Ramakrishnan [10]. The concept of matrices $\bar{G}_a$ "conjugate" to G_a as used here considerably clarifies the nature of the building up process.

7. Transformation Laws

We now consider in more detail the nature of the 2^ν-dimensional representations of $SO(M, C)$. Write

$$S = \exp \tfrac{1}{4}\lambda_{ab}G_{ab},$$
$$\bar{S} = \exp \tfrac{1}{4}\lambda_{ab}\bar{G}_{ab}, \tag{7.1}$$

for the transformation matrices of the two (inequivalent) representations. The parameters λ_{ab} are complex. We will consider the infinitesimal forms of these matrices.

From (5.24) with $(k, l) = (1, 2)$ or $(2, 1)$, we obtain

$$\begin{aligned} G_a\bar{G}_{bc} &= G_{abc} + \delta_{ab}G_c - \delta_{ac}G_b, \\ G_{bc}G_a &= G_{bca} - \delta_{ab}G_c + \delta_{ac}G_b, \\ \bar{G}_aG_{bc} &= \bar{G}_{abc} + \delta_{ab}\bar{G}_c - \delta_{ac}\bar{G}_b, \\ \bar{G}_{bc}\bar{G}_a &= \bar{G}_{bca} - \delta_{ab}\bar{G}_c + \delta_{ac}\bar{G}_b, \end{aligned} \tag{7.2}$$

whence

$$\begin{aligned} \tfrac{1}{2}(G_{bc}G_a - G_a\bar{G}_{bc}) &= \delta_{ac}G_b - \delta_{ab}G_c, \\ \tfrac{1}{2}(\bar{G}_{bc}\bar{G}_a - \bar{G}_aG_{bc}) &= \delta_{ac}\bar{G}_b - \delta_{ab}\bar{G}_c. \end{aligned} \tag{7.3}$$

These are the infinitesimal forms of the statements

$$\begin{aligned} G_a &= O_{ab}SG_b\bar{S}^{-1}, \\ \bar{G}_a &= O_{ab}\bar{S}G_bS^{-1}, \end{aligned} \tag{7.4}$$

where O_{ab} is a general element of $SO(M, C)$ $(O_{ab}O_{cb} = \delta_{ac})$ and S and $\bar{S}$ are the matrices that represent it. Equation (7.4) can be regarded as transformation laws for G_a and $\bar{G}_a$ under which their components remain invariant. It then follows immediately that if we subject a 2×2 matrix to the transformations

$$\begin{array}{llll} \text{(a)} & A \to SAS^{-1} & \text{(kind 1)}, \\ \text{(b)} & A \to SA\bar{S}^{-1} & \text{(kind 2)}, \\ \text{(c)} & A \to \bar{S}A\bar{S}^{-1} & \text{(kind 3)}, \\ \text{(d)} & A \to \bar{S}AS^{-1} & \text{(kind 4)}, \end{array} \tag{7.5}$$

then,

(a) The components referred to $\mathscr{G}^{(1)}$ of the quantity $A_A{}^B$ of kind 1,

$$a, a_{ab}, a_{abcd}, \ldots, a_{a_1 \cdots a_\sigma}, \tag{7.6}$$

($\sigma = \mu$ or ν according as $\mu = \nu + 1$ is even or odd) transform as *tensors* of $SO(M, C)$. For μ even the tensor of greatest rank in this set is self-dual,

$$a_{a_1 \cdots a_\mu} = a^D_{a_1 \cdots a_\mu}.$$

(b) The components referred $\mathscr{G}^{(2)}$, of the quantity $A_{A\bar{B}}$ of kind 2,

$$a_a, a_{abc}, \ldots, a_{a_1 \cdots a_\sigma}, \tag{7.7}$$

($\sigma = \mu$ or ν according as $\mu = \nu + 1$ is odd or even) are $SO(M, C)$ tensors. For μ odd, $a_{a_1 \cdots a_\mu} = a^D_{a_1 \cdots a_\mu}$.

(c) The components referred to $\bar{\mathscr{G}}^{(1)}$, of the quantity $A_{\bar{B}}{}^{\bar{A}}$ of kind 3,

$$a, a_{ab}, a_{abcd} \cdots a_{a_1 \cdots a_\sigma} \tag{7.8}$$

($\sigma = \mu$ or ν for μ even or odd) are $SO(M, C)$ tensors. For μ even,

$$a_{a_1 \cdots a_\mu} = - a^D_{a_1 \cdots a_\mu}.$$

(d) The components referred to $\bar{\mathscr{G}}^{(2)}$, of the quantity $A^{\bar{A}B}$ of kind 4,

$$a_a, a_{abc}, \ldots, a_{a_1 \cdots a_\sigma}, \tag{7.9}$$

($\sigma = \mu$ or ν for μ odd or even) are $SO(M, C)$ tensors. For μ odd,

$$a_{a_1 \cdots a_\mu} = - a^D_{a_1 \cdots a_\mu}.$$

We define 2^ν component spinors of $SO(M, C)$ as follows:

$$
\begin{aligned}
&(1)\quad \varphi_A \text{ with transformation law } \varphi \to S\varphi,\\
&(2)\quad \chi_{\bar{A}} \text{ with transformation law } \chi \to \bar{S}\chi,\\
&(3)\quad \bar{\varphi}^A \text{ with transformation law } \bar{\varphi} \to \bar{\varphi}S^{-1},\\
&(4)\quad \bar{\chi}_{\bar{A}} \text{ with transformation law } \bar{\chi} \to \bar{\chi}\bar{S}^{-1}.
\end{aligned}
\tag{7.10}
$$

Then,

$$
\begin{aligned}
&\bar{\varphi}\mathscr{G}^{(1)}\varphi \text{ is a set of \textit{even} rank tensors up to rank } \mu.\\
&\qquad \text{If } \mu \text{ is even the tensor of rank } \mu \text{ is \textit{self-dual}.}\\
&\bar{\varphi}\mathscr{G}^{(2)}\chi \text{ is a set of \textit{odd} rank tensors up to rank } \mu.\\
&\qquad \text{If } \mu \text{ is odd the tensor of rank } \mu \text{ is \textit{self-dual}.}\\
&\bar{\chi}\bar{\mathscr{G}}^{(1)}\chi \text{ is a set of \textit{even} rank tensors up to rank } \mu.\\
&\qquad \text{If } \mu \text{ is even the tensor of rank } \mu \text{ is \textit{anti-selfdual}.}\\
&\bar{\chi}\bar{\mathscr{G}}^{(2)}\varphi \text{ is a set of \textit{odd} rank tensors up to rank } \mu.\\
&\qquad \text{If } \mu \text{ is odd the tensor of rank } \mu \text{ is \textit{anti-selfdual}.}
\end{aligned}
\tag{7.11}
$$

If we restrict our considerations to the subgroup $SO(2\nu, C)$, S and $\bar{S}$ become identical, and we can use the $\mathscr{V}(2\nu)$ quantity of Section 3 as basis. In this case we just have the result that

$$\text{For } SO(2\nu, C), \bar{\varphi}\mathscr{G}\varphi \text{ is a set of tensors of rank } p, \quad (p = 0, 1, 2, \ldots 2\nu). \tag{7.12}$$

8. The Spinor Metric

Since there is only one 2^ν-dimensional irreducible representation of $C_{2\nu}$, the sets of matrices G_α, G_α^T and $-G_\alpha^T$ $(\alpha = 1,\ldots, 2\nu)$ are equivalent. (The superscript T denotes transpose.) Thus we can define a matrix D satisfying

$$D^{-1}G_\alpha D = (-)^\nu G_\alpha^T \qquad (\alpha = 1,\ldots, 2\nu), \tag{8.1}$$

the factor being chosen so that G_N satisfies the same relation

$$D^{-1}G_N D = (-)^\nu G_N^T. \tag{8.2}$$

Extending to M dimensions, these equations become

$$
\begin{aligned}
D^{-1}G_a D &= G_a^T \qquad &(a = 1,\ldots, M) \qquad &\text{for } \nu \text{ even},\\
D^{-1}G_a D &= -\bar{G}_a^T \qquad &(a = 1,\ldots, M) \qquad &\text{for } \nu \text{ odd}.
\end{aligned}
\tag{8.3}
$$

From (8.1) we can show that $D^T D^{-1}$ commutes with all the generators and is therefore a multiple λ of the unit matrix. Then $D^T = \lambda D$, $D = \lambda D^T$, $D = \lambda^2 D$ and therefore $\lambda^2 = 1$, $\lambda = \pm 1$. Thus D is either symmetric or skewsymmetric:

$$D^T = \lambda D \qquad (\lambda = \pm 1). \tag{8.4}$$

We also assert that D can be chosen to satisfy $D^2 = \lambda$. This will be proved in the next section.

In order to fix λ exactly, we proceed as follows: From (8.1),

$$G_{\alpha_1 \cdots \alpha_p} D = (-)^{p\nu+\kappa} \lambda [G_{\alpha_1 \cdots \alpha_p} D]^T, \tag{8.5}$$

where $2\kappa = p$ or $p - 1$ according as p is even or odd. Therefore, the matrices $G_{\alpha_1 \cdots \alpha_p} D$ have the same symmetry as D or the opposite symmetry according as $p\nu + \kappa$ is even or odd. Thus if we subtract the number of matrices $G_{\alpha_1 \cdots \alpha_p} D$ with the symmetry opposite to that of D from the number with the same symmetry as D, we get the number

$$S_\nu = \sum_{p=0}^{2\nu} (-)^{p\nu+\kappa} \binom{2\nu}{p}. \tag{8.6}$$

Since there are $\frac{1}{2} \cdot 2^\nu (2^\nu + 1)$ symmetric and $\frac{1}{2} 2^\nu \cdot (2^\nu - 1)$ skew-symmetric $2^\nu \times 2^\nu$ matrices, and since the $G_{\alpha_1 \cdots \alpha_p} D$ are linearly independent, the number S will be 2^ν or -2^ν according as D is symmetric or skewsymmetric.

Hence

$$\lambda = \frac{1}{2^\nu} S_\nu = \frac{1}{2^\nu} \sum_p (-)^{p\nu+\kappa} \binom{2\nu}{p}. \tag{8.7}$$

We next show that S_ν satisfies

$$S_{\nu+1} = 2(-)^\nu S_\nu. \tag{8.8}$$

We have

$$\begin{aligned} S_{\nu+1} &= \sum_p (-)^p (-)^{p\nu+\kappa} \binom{2\nu+2}{p} \\ &= \sum_p (-)^p (-)^{p\nu+\kappa} \left[\binom{2\nu}{p-2} + 2\binom{2\nu}{p-1} + \binom{2\nu}{p} \right] \\ &= \sum_p \binom{2}{p} [(-)^{p+2} (-)^{(p+2)\nu+\kappa_{p+2}} + 2(-)^{p+1} (-)^{(p+1)\nu+\kappa_{p+1}} + (-)^p (-)^{p\nu+\kappa}] \end{aligned}$$

(where $2\kappa_{p+1} = p + 1$ or p according as $p + 1$ is even or odd, $2\kappa_{p+2} = p + 2$ or $p + 1$ according as $p + 2$ is even or odd). Now, it is easily verified that $(-)^{\kappa_{p+2}} = -(-)^{\kappa}$ and that $(-)^{\kappa_{p+1}} = -(-)^{p+\kappa}$ so that

$$S_{\nu+1} = \sum_p \binom{2\nu}{p} [2(-)^{\nu} (-)^{p\nu+\kappa}] = 2(-)^{\nu} S_{\nu} .$$

Hence, defining $2\rho = \nu$ or $\nu - 1$ according as ν is even or odd, we can easily show by induction that

$$S_{\nu} = 2^{\nu}(-)^{\rho+\nu}. \tag{8.9}$$

The symmetry or skewsymmetry of D is then given by

$$D^T = \lambda D, \qquad \lambda = (-)^{\rho+\nu}, \tag{8.10}$$

where $2\rho = \nu$ or $\nu - 1$ according as ν is even or odd. The following table gives the values of λ for various dimensionalities

TABLE II

2ν	2	4	6	8	10	12
λ	$-$	$-$	$+$	$+$	$-$	$-$

Note the well-known result for C_2 and C_4 the spin metric is skewsymmetric.

Now we go back to (8.3) and consider the properties of D that relate to the representation theory of $SO(M, C)$. The two cases ν odd and ν even must be carefully distinguished.

(a) *ν odd.* We can show that

$$\begin{aligned} G_{ab}D + DG_{ab}^T &= 0, \\ \bar{G}_{ab}D + D\bar{G}_{ab}^T &= 0, \end{aligned} \tag{8.11}$$

which are the infinitesimal forms of

$$\begin{aligned} SDS^T &= D, \\ \bar{S}D\bar{S}^T &= D, \\ S^{-T}DS^{-1} &= D, \\ \bar{S}^{-T}D\bar{S}^{-1} &= D. \end{aligned} \tag{8.12}$$

These relations can be considered to be four alternative $SO(M, C)$ transformation laws of the matrix D, under which its components remain unchanged. They correspond to the assignments

$$D_{AB}\,,\, D^{\bar{A}\bar{B}},\, D^{AB} \qquad \text{and} \qquad D_{\bar{A}\bar{B}} \tag{8.13}$$

of spinor indices, respectively [see (7.6)]. Thus D can be used as a "raising and lowering" operator for spinor indices (care needs to be taken in the D skewsymmetric cases). Thus the representations S and $(S^T)^{-1}$ are equivalent. For example, with the aid of the matrix D we can construct sets of tensors in M dimensions as bilinears formed from a *single* spinor [Cf. (7.7) constructed from a pair of spinors]:

$$\varphi D\mathscr{G}^{(1)}\varphi, \qquad \text{and} \qquad \chi D\bar{\mathscr{G}}^{(1)}\chi, \tag{8.14}$$

transform as sets of even rank tensors under $SO(M, C)$. To make this more precise, note that $D\mathscr{G}^{(1)}$ and $D\bar{\mathscr{G}}^{(1)}$ are sets of $2^\nu \times 2^\nu$ matrices, and we can separate the symmetric ones from the skewsymmetric ones, this separation being invariant under $SO(M, C)$. That is,

$$DG_{a_1\cdots a_p} \qquad (p \text{ even}) \tag{8.15}$$

is either symmetric or skewsymmetric. The particular symmetry property can be obtained from (8.3) in the form

$$\begin{aligned} DG_{a_1\cdots a_p} &= (-)^\kappa\, \lambda[DG_{a_1\cdots a_p}]^T, \\ DG_{a_1\cdots a_p} &= -\,(-)^{\kappa+\rho}\,[DG_{a_1\cdots a_p}]^T. \end{aligned} \tag{8.16}$$

The symmetry properties of $DG_{a_1\cdots a_p}$ (p even) given by this formula are tabulated in Table III.

TABLE III

Correspondence Between a Rank 2 Spinor and Sets of p-Forms for $SO(2\nu + 2, C)$

		Symmetric	Skewsymmetric
ν odd	ρ even	$p = 2, 6, 10,\ldots$	$p = 0, 4, 6,\ldots$
	ρ odd	$p = 0, 4, 8,\ldots$	$p = 2, 6, 10,\ldots$

These results, combined with (5.17) enable us to obtain, for any dimensionality M for which ν is odd, the correspondences that exist between rank 2 spinors and sets of skewsymmetric tensors. For example, for $\nu = 3$ we have $\rho = 1$ and $M = 8$. A *symmetric* rank two spinor (8×8 matrix) has components which transform as a scalar and a rank 4 self-dual symmetric tensor. (We may check that the number of components correspond to $\frac{1}{2}(8.9) = 1 + (\frac{1}{2})\binom{8}{4})$. A skewsymmetric rank 2 spinor corresponds to a rank 2 skewsymmetric tensor. Table III also applies to the $D\bar{G}_{a_1 \cdots a_p}$. Note that in (8.14) only the *symmetric* parts of $D\mathscr{G}^{(1)}$ or $D\bar{\mathscr{G}}^{(1)}$ contributes to the expressions and the table allows us to determine which particular tensor ranks occur among the bilinears for any particular dimensionality.

(b) *ν even.* When ν is even, (8.11) is replaced by

$$\begin{aligned} G_{ab}D + D\bar{G}_{ab}^T &= 0, \\ \bar{G}_{ab}D + DG_{ab}^T &= 0, \end{aligned} \tag{8.17}$$

which shows that D has the following transformation laws under which it is invariant,

$$\begin{aligned} SD\bar{S}^T &= D, \\ \bar{S}DS^T &= D, \\ S^{-T}D\bar{S}^{-1} &= D, \\ \bar{S}^{-T}DS^{-1} &= D, \end{aligned} \tag{8.18}$$

corresponding to the assignments

$$D_A{}^{\bar{B}},\ D^{\bar{A}}{}_B,\ D^A{}_{\bar{B}},\ D_{\bar{A}}{}^B. \tag{8.19}$$

Thus D is no longer a raising and lowering operator, but instead converts barred spinor indices to unbarred and vice versa ($\bar{S}$ is equivalent to $(S^T)^{-1}$). Instead of (8.15), the bilinears that can be constructed from a single spinor are

$$\varphi D\bar{\mathscr{G}}^{(2)}\varphi \qquad \text{and} \qquad \chi D\mathscr{G}^{(2)}\chi, \tag{8.20}$$

which contain only tensors of *odd* rank. In the present case we are interested in the symmetry properties of $DG_{a_1 \cdots a_p}$ (*p odd*) and these are obtained from $D^2 = \lambda$ and (8.3),

$$DG_{a_1 \cdots a_p} = (-)^{\kappa+\rho}\,[DG_{a_1 \cdots a_p}]^T, \tag{8.21}$$

which gives Table IV

TABLE IV

Correspondence Between a Rank 2 Spinor and Sets of p-Forms for $SO(2\nu + 2, C)$

		Symmetric	Skewsymmetric
ν even	ρ odd	$p = 3, 7, 11,\ldots$	$p = 1, 5, 9,\ldots$
	ρ even	$p = 1, 5, 9,\ldots$	$p = 3, 7, 11,\ldots$

As an example, consider the case $\nu = 2$. Then $\rho = 1$ and $M = 6$. A rank 2 spinor with its two spinor indices of the same type can be regarded as a set of tensors of odd rank. Consulting Table IV, and (5.20) we see that if it is *symmetric* its components behave as a rank 3 self-dual skewsymmetric tensor (check $(\frac{1}{2})\, 4.5 = (\frac{1}{2})\binom{6}{3}$) if *skewsymmetric*, they behave as a *six vector* under $SO(6, C)$ (check $(\frac{1}{2})\, 4.3 = 6$).

The information in Tables III and IV refers to those rank 2 spinors with two indices of the same type, so that the decomposition into a symmetric and skewsymmetric part is an invariant process. The analogous correspondence between rank two spinors with *mixed* indices has already been dealt with: $A_A{}^B$ and $A_{\bar{A}}{}^{\bar{B}}$ are of "kind 1" and "kind 3" and correspond to a set of tensors of ranks 0, 2, 4, 6,..., while $A_{A\bar{B}}$ and $A^{\bar{A}B}$ are of "kind 2" and "kind 3" and correspond to tensors of ranks 1, 3, 5,... . The remaining cases $A_A{}^{\bar{B}}$ and $A_{\bar{A}}{}^B$ can be obtained by multiplication by D from $A_{A\bar{B}}$ if ν is odd and from $A_A{}^B$ if ν is even and so correspond to tensors of rank 1, 3, 5,... or 0, 2, 4,... according as ν is odd or even.

9. Construction of the Matrix D

We have already noted that if G_a $(a = 1, M)$ $(M = 2\nu + 2)$ are the Hermitian generators of the irreducible representation of $C_{2\nu}$ together with the two extra matrices previously discussed, then

$$F_a = \begin{pmatrix} & G_a \\ \bar{G}_a & \end{pmatrix} \tag{9.1}$$

are the generators of the irreducible representation of C_M. Suppose D is the "spinor metric" of $C_{2\nu}$ for the representation G_a. If ν is *even* ($\mu = M/2$ odd), then $D^{-1}G_aD = G_a{}^T$, $D^{-1}\bar{G}_aD = \bar{G}_a{}^T$, therefore *for ν even*

$$\mathcal{D} = \begin{pmatrix} & D \\ -D & \end{pmatrix} \tag{9.2}$$

satisfies

$$\mathcal{D}^{-1}F_a\mathcal{D} = (-)^{\mu} F_a{}^T, \tag{9.3}$$

and so by (8.1) is the spin metric for the representation F_a of C_M. On the other hand, if ν is *odd* (μ even) we have $D^{-1}G_aD = -\bar{G}_a{}^T$, $D^{-1}\bar{G}_aD = -G_a{}^T$. Then *for ν odd*

$$\mathcal{D} = \begin{pmatrix} D & \\ & -D \end{pmatrix} \tag{9.4}$$

satisfies $D^{-1}F_aD = (-)^\mu F_a{}^T$ and by (8.1) is the spin metric for C_M. We therefore have a method of constructing a hierarchy of spin metrics along with the building up of the representations of even Clifford algebras given in Section 6. Starting from the trivial $D = 1$ of C_0, we find for C_2,

$$D = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \tag{9.5}$$

which is the well-known "ϵ" [1, 5, 11] of two-component spinor algebra. Then for C_4 we get

$$D = \begin{pmatrix} \epsilon & \\ & -\epsilon \end{pmatrix}, \tag{9.6}$$

which is the "charge conjugation matrix", or Pauli matrix C [5, 11] in the Weyl representation of the Dirac algebra. For C_6 we have the first *symmetric* spinor metric $\begin{pmatrix} & C \\ -C & \end{pmatrix}$ and so on. From the obvious consequence of (9.2) and (9.4), $D^2_{(\nu-1)} = -(-)^\nu D^2_{(\nu)}$ we can prove by induction that $D^2_{(\nu)} = (-)^{\nu+\rho} = \lambda$, which is the relation mentioned without proof in the previous section.

10. Explicit Construction of Transformation Matrices: Reflection Properties

The building up process given in Section 6 can be applied to the representations of orthogonal groups. Consider

$$F_a = \begin{pmatrix} & G_a \\ \bar{G}_a & \end{pmatrix}, \qquad F_{M+1} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad F_{M+2} = -i, \tag{10.1}$$

where G_a are $2^\nu \times 2^\nu$ matrices associated with $C_{2\nu}$ $(M = 2\nu + 2)$. The generators of the spinor representation of $SO(M, C)$ are then

$$\begin{aligned} &\tfrac{1}{2}F_{ab}, \qquad (a, b = 1, M+2),\\ &\tfrac{1}{2}\bar{F}_{ab}, \end{aligned}$$

where

$$\begin{aligned} F_{ab} = \bar{F}_{ab} &= \begin{pmatrix} G_{ab} & \\ & \bar{G}_{ab} \end{pmatrix} \qquad (a, b = 1, M),\\ F_{a,M+1} = \bar{F}_{a,M+1} &= \begin{pmatrix} & -G_a \\ \bar{G}_a & \end{pmatrix},\\ F_{a,M+2} = -\bar{F}_{a,M+2} &= \begin{pmatrix} & iG_a \\ i\bar{G}_a & \end{pmatrix},\\ F_{M+1,M+2} = -\bar{F}_{M+1,M+2} &= \begin{pmatrix} i & \\ & -i \end{pmatrix}. \end{aligned} \tag{10.2}$$

We then see, that if $SO(M, C)$ is restricted to the subgroup $SO(M, C)$, the representations S and $\bar{S}$ of $SO(M+2, C)$ become equivalent and decompose according to

$$S \to \begin{pmatrix} S & \\ & \bar{S} \end{pmatrix}, \qquad \bar{S} \to \begin{pmatrix} S & \\ & \bar{S} \end{pmatrix}, \tag{10.3}$$

where the two parts in the direct sum are the two different spinor representations of $SO(M, C)$. Thus, under the $SO(M, C)$ subgroup, the spinor ψ_A of $SO(M+2, C)$ behaves like two separate spinors of $SO(M, C)$:

$$\psi_A = \begin{pmatrix} \varphi_A \\ \chi^{\bar{A}} \end{pmatrix}. \tag{10.4}$$

Now consider the $SO(M+2, C)$ rotation in the $(M, M+1)$ plane. $F_{M,M+1} = \left(\begin{smallmatrix} & -i \\ -i & \end{smallmatrix}\right)$ so that $\exp i(\pi/2) F_{M,M+1}$ is just $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$ and $\exp i(\pi/2) \bar{F}_{M,M+1}$ is also $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$. These represent rotations through angle π in the $(M, M+1)$ plane. We consider the subgroup of $SO(M+2, C)$ consisting of $SO(M, C)$ rotations together with the rotation through π in the $(M, M+1)$ plane. This subgroup has the representation

$$S = \bar{S} = \begin{pmatrix} S & \\ & \bar{S} \end{pmatrix}, \qquad B = \bar{B} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \tag{10.5}$$

The rotation through π in the $(M, M+1)$ plane just reverses the components v_M, v_{M+1} of a vector v_a $(a = 1, M+2)$ and so, for the M-dimensional subspace, corresponds to a *reflection*; so the subgroup represented by (10.5) is isomorphic to $O(M, C)$. The quantity (10.4) is then a representation of the *general* orthogonal group in M dimensions. The $v_a \to v_a$ $(a = 1, \cdots, M-1)$, $v_M \to -v_M$ reflection is then represented in spinor space by

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \to \begin{pmatrix} \chi \\ \varphi \end{pmatrix}. \tag{10.6}$$

11. A Particular Case

In this section we indicate briefly how the usual algebra of two-component and four-component spinors can be dealt with as a particular case of the generalized theory we have presented.

The group $SO(6, C)$ has spinor representation matrices S (and $\bar{S}$) that belong to $SL(4, C)$. The number of complex parameters for both these groups is 15, so that the group of orthogonal matrices is mapped onto the *whole* of $SL(4, C)$, rather than a subgroup. The mapping is 4 to 1 since (7.4), which can be considered as defining relation for S and $\bar{S}$, is unchanged by multiplication of S and $\bar{S}^{-1}$ by the same numerical factor. If we impose unimodularity this factor can only be ± 1 or $\pm i$, and then up to these phase factors S and $\bar{S}$ are determined uniquely by (7.4). Thus $SO(6, C)$ is isomorphic to $SL(4, C)/C4$ where $C4$ is the cyclic group of order 4. (Similarly the real subgroup $SO(6)$ of $SO(6, C)$ can be shown to be isomorphic to $SU(4)/C2$).

The considerations of the previous section for the case $M = 4$ lead to properties of the $O(4, C)$ subgroup of $SO(6, C)$. Recalling the example at the end of Section 8, a 4×4 matrix A_{AB} with $SO(6, C)$ transformation law

$$A \to SA\bar{S}^{-1}$$

can be written

$$A = v_a F_a + \frac{1}{12} v_{abc} F_{abc} \qquad (a, b = 1,..., 6) \tag{11.1}$$

($Dv_a F_a$ and $Dv_{abc} F_{abc}$ being, respectively, skewsymmetric and symmetric). The tensor v_{abc} is self-dual, so that

$$A = v_6 F_6 + v_{a56} F_{a56} + \frac{1}{12} v_{ab6} F_{ab6} + v_a F_a + v_5 F_5 \qquad (a, b = 1,..., 4) \tag{11.2}$$

and, recalling the paragraph following (6.10),

$$A = -iv_6 - iv_{a56}\gamma_a + \tfrac{1}{2} iv_{ab6}\gamma_{[a}\gamma_{b]} + v_a\gamma_a\gamma_5 + v_5\gamma_5 \qquad (a, b = 1,..., 4) \tag{11.3}$$

in terms of the Weyl representation of the Dirac algebra.

Defining

$$X = -iv_6\,, \qquad X_a = -iv_{a56}\,, \qquad X_{ab} = iv_{ab6}\,,$$
$$X_{a5} = v_a\,, \qquad X_5 = v_5 \qquad (a, b = 1,..., 4),$$

we can write

$$A = X + X_a\gamma_a + \tfrac{1}{2} X_{ab}\gamma_{[a}\gamma_{b]} + X_{a5}\gamma_{a5} + X_5\gamma_5 \qquad (a = 1,..., 4). \tag{11.4}$$

By the way the component set X has been defined, we see that, under rotations through π in the (4, 5) plane,

$$X, X_4\,, X_{ab}\,, X_{a5} \qquad (a, b = 1, 2, 3)$$

are unchanged, while

$$X_a\,, X_{a4}\,, X_{45}\,, X_5 \qquad (a = 1, 2, 3)$$

are multiplied by -1. Therefore, under the $O(4, C)$ subgroup of $SO(6, C)$ the six-vector and self-dual rank 3 tensor of (11.1) behave as a scalar, a vector, a rank 2 skewsymmetric tensor, a pseudovector and a pseudoscalar (compare this with the statement (7.12) for $SO(4, C)$). The spin transformation (10.6) corresponds to the "parity" transformation $X_a \to -X_a$ $(a = 1, 2, 3)$ $X_4 \to X_4$.

We get precisely the same structure under $O(4, C)$ if we start with an $SO(6, C)$ spinor $A_A{}^B$ with transformation law $A \to SAS^{-1}$;

$$A = v + \tfrac{1}{2}\, v_{ab} F_{ab} \qquad (a, b = 1,..., 6) \tag{11.5}$$

leads to the expression (11.4) with the same $O(4, C)$ transformation properties for the set X, X_a, X_{ab}, X_{a5}, X_5 $(a, b = 1, 4)$. This is to be expected since S and $\bar{S}$ are not distinguished for the $O(4, C)$ subgroup.

Consider now the subgroup of $SO(6, C)$ for which the spinor representation S satisfies

$$S^+BS = B, \tag{11.6}$$

with B given by (10.5). Such an S clearly belongs to the $SU(2, 2)$ subgroup of $SL(4, C)$. It is generated by $\frac{1}{2} E_{ab}$ $(a, b = 1,... 6)$ with *real* parameters, where

$$E_{ab} = F_{ab}\,, \qquad E_{a4} = iF_{a4}\,, \qquad E_{a5} = iF_{a5} \qquad (a = 1, 2, 3, 6). \tag{11.7}$$

The required subgroup of $SO(6, C)$ is therefore the *real* pseudo-orthogonal group $SO(4, 2)$. The subgroup of $SO(4, 2)$ corresponding to the $O(4, C)$ subgroup of $SO(6, C)$ is just the Lorentz group $O(3, 1)$. For $SO(4, 2)$ we can construct the following covariant bilinears from a spinor ψ:

$$\begin{gathered} \psi^+B\psi \text{ (a scalar)}, \\ \psi^+BE_{ab}\psi \qquad (a, b = 1,..., 6) \text{ (a rank 2 tensor)}. \end{gathered} \tag{11.8}$$

The tensor decomposes under the Lorentz subgroup $O(3, 1)$ to give just the usual Dirac bilinears.

The transformation matrix for ψ, for the Lorentz subgroup $SO(3, 1)$, has the form

$$L = \begin{pmatrix} S & \\ & \bar{S} \end{pmatrix}, \tag{11.9}$$

where the 2×2 matrix S is generated by $\frac{1}{2} e_{ab}$ $(a, b = 1, \ldots 4)$ and $\bar{S}$ by $\frac{1}{2} \bar{e}_{ab}$ with *real* parameters;

$$\begin{aligned} e_{ab} &= G_{ab}, & e_{a4} &= iG_{a4}, \\ \bar{e}_{ab} &= \bar{G}_{ab}, & \bar{e}_{a4} &= i\bar{G}_{a4}, \end{aligned} \qquad (a, b = 1, 2, 3), \tag{11.10}$$

with G_{ab}, $\bar{G}_{ab}$ $(a, b = 1, \ldots 4)$ the generators of the spin representations of $SO(4, C)$. Combining (11.6) with (11.9) $(L^{+}BL = B)$ shows that the 2×2 representations S and $\bar{S}$ of $SO(3, 1)$ satisfy $\bar{S} = (S^{+})^{-1}$. Therefore

$$L = \begin{pmatrix} S & \\ & (S^{\dagger})^{-1} \end{pmatrix} \tag{11.11}$$

—just the usual Lorentz transformation matrix for Dirac spinors in the Weyl representation.

The self-duality of $G_{ab}(G_{ab} = \frac{1}{2} \epsilon_{abcd} G_{cd})$ (which implies that the 2-component spinors representations of $SO(4, C)$ are not faithful) leads to

$$e_{12} = ie_{43}, \qquad e_{23} = ie_{41}, \qquad e_{31} = ie_{42}, \tag{11.12}$$

so that since S has real parameters for the Lorentz subgroup $SO(3, 1)$ of $SO(4, C)$, it *is* a faithful representation for that subgroup.

In terms of the Pauli matrices,

$$e_{ab} = i\epsilon_{abc}\sigma_c, \qquad e_{4a} = \sigma_a \qquad (a, b, c = 1, 2, 3). \tag{11.13}$$

The group of matrices S corresponding to $SO(3, 1)$ is just $SL(2, C)$.

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