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Algebraic Properties of Relativistic Equations for Zero Rest-Mass

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Abstract

Some algebraic properties of the generalised Pauli matrices of Barut, Muzinich and Williams are derived and used to demonstrate the equivalence of the zero-mass equations of Nelson and Good with those of Dirac, Fierz and Pauli. The conserved rank four tensor of the spin-2 theory is shown to have the structure of Bel's tensor for a gravitational field satisfying Einstein's empty space equations, in the linearised version of general relativity.

1. *Introduction*

Recently, Nelson & Good (1969) have given a new description of massless spin-j particles in terms of the relativistic wave equation

$$\bar{s}^{\mu_1\mu_2\cdots\mu_{2j}}\,\partial\Phi/\partial x^{\mu} = 0 \tag{1.1}$$

where Φ is a $(2j+1)$-component quantity transforming according to the $(j,0)$ representation of the inhomogeneous proper Lorentz group (L) and the $\bar{s}^{\mu_1\cdots\mu_{2j}}$ are a set of $(2j+1)\times(2j+1)$ matrices, completely symmetric in the tensor index set and traceless in the sense that

$$g_{\mu\nu}\, s^{\mu\nu\rho_3\cdots\rho_{2j}} = 0 \tag{1.2}$$

($g_{\mu\nu}$ is the flat-space metric with non-vanishing components $-g_{11}=-g_{22}=-g_{33}=g_{44}=1$, which will be used as raising and lowering operator for tensor indices). Let the matrix for the $(j,0)$ representation of an element Λ of L be denoted by $\mathscr{D}^{(j)}[\Lambda]$. The infinitesimal generators $M_{\mu\nu}=-M_{\nu\mu}$ can be taken to be

$$\begin{aligned}(M^{23},M^{31},M^{12}) &= \mathbf{s}\\ (M^{14},M^{24},M^{34}) &= i\mathbf{s}\end{aligned} \tag{1.3}$$

The three s being the Hermitian generators of the spin-j representation of the three-dimensional rotation subgroup (Brink & Satchler, 1962),

$$\begin{aligned}[s^3\Phi]_m &= m\Phi_m \qquad (m=-j,-j+1,\cdots j)\\ [s^{\pm}\Phi]_{m+1} &= \{(j\mp m)(j\pm m+1)\}^{1/2}\Phi_m \qquad (s^{\pm}=s^1\pm is^2)\end{aligned} \tag{1.4}$$

The $(0,j)$ representation is given by $\bar{\mathscr{D}}^{(j)} = (\mathscr{D}^{(j)\dagger})^{-1}$ and the most general representation of L is (j,k) given by $\mathscr{D}^{(j)} \otimes \bar{\mathscr{D}}^{(k)}$ (Corson, 1953).

The covariance of (1.1) under $x^\mu \to \Lambda_\nu{}^\mu x^\nu$, $\Phi \to \mathscr{D}^{(j)}[\Lambda]\Phi$ is ensured by requiring

$$\bar{\mathscr{D}}^{(j)}[\Lambda]\,\bar{s}^{\mu_1\cdots\mu_{2j}}\,\bar{\mathscr{D}}^{(j)\dagger}[\Lambda] = (\Lambda^{-1})^{\mu_1}_{\nu_1}\cdots(\Lambda^{-1})^{\mu_{2j}}_{\nu_{2j}}\,\bar{s}^{\nu_1\cdots\nu_{2j}}. \tag{1.5}$$

As shown by Nelson and Good, this requirement is sufficient to determine the $\bar{s}^{\mu_1\cdots\mu_{2j}}$ uniquely, to within an overall numerical factor. The quantities $\bar{s}^{\mu_1\cdots\mu_{2j}}$ and the relation (1.5), together with associated quantities $s^{\mu_1\cdots\mu_{2j}}$ defined by

$$\mathscr{D}^{(j)}[\Lambda]\,s^{\mu_1\cdots\mu_{2j}}\,\mathscr{D}^{(j)\dagger}[\Lambda] = (\Lambda^{-1})^{\mu_1}_{\nu_1}\cdots(\Lambda^{-1})^{\mu_{2j}}_{\nu_{2j}}\,s^{\nu_1\cdots\nu_{2j}} \tag{1.5'}$$

were first introduced by Barut *et al.* (1963). They have been discussed by Weinberg (1964) and, in connection with formulations of wave equations for particles with non-vanishing mass, by Williams (1964) and by Sankaranarayanan & Good (1965). In Barnet et al. (1963) they are constructed from direct products of Pauli matrices by using Clebsch–Gordon coefficients to pick out the spin-j part from the sets of two-fold (spin-$\frac{1}{2}$) spinor indices occurring in the direct products. The method described below is simpler in that the Clebsch–Gordon coefficients are not used explicitly. This is possible because we shall use the characterisation of $(j,0)$ as a quantity with a completely symmetric set of two-fold spinor indices, rather than as a $(2j+1)$-component column as implied by (1.3) and (1.4). Let $D[\Lambda]$ be the $(\frac{1}{2},0)$ representation of L and consider a quantity

$$\phi_{A_1\ldots A_{2j}}$$

completely symmetric in its two-fold indices and transforming according to

$$\phi_{A_1\cdots A_{2j}} \to D^{B_1}_{A_1}\cdots D^{B_{2j}}_{A_{2j}}\,\phi_{B_1\cdots B_{2j}} \tag{1.6}$$

Because of the symmetry, a particular component will be specified by the number of 1's and the number of 2's in its index set (n_1 and n_2 with $n_1+n_2 = n_1+n_2$) and can be written $\phi(n_1 n_2)$. Writing $2m = n_1 - n_2$, the $2j+1$ component column

$$\Phi_m = \binom{2j}{j+m}^{1/2}\phi(n_1 n_2) \qquad (m = -j, -j+1, \ldots j) \tag{1.7}$$

will have the $\mathscr{D}^{(j)}$ transformation law defined by (1.3) and (1.4). The proof is straightforward so we shall omit it.

2. *Construction of Generalised Pauli Matrices*

Denoting the three Pauli matrices by $\boldsymbol{\sigma}$ and the spinor metric $(_{-1}{}^{1})$ by ϵ, we define

$$\left.\begin{aligned} \sigma^\mu &= (\boldsymbol{\sigma}, 1) \\ \bar{\sigma}^\mu &= \epsilon(\sigma^\mu)^T\epsilon^{-1} = (-\boldsymbol{\sigma}, 1) \end{aligned}\right\} \tag{2.1}$$

which satisfy

$$\left.\begin{aligned} \tfrac{1}{2}(\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu) = g_{\mu\nu}.1 \\ \tfrac{1}{2}(\bar{\sigma}_\mu \sigma_\nu + \bar{\sigma}_\nu \sigma_\mu) = g_{\mu\nu}.1 \end{aligned}\right\} \tag{2.2}$$

and we define

$$\sigma^{\mu\nu} = \tfrac{1}{2}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$$

then

$$\bar{\sigma}^{\mu\nu} = -(\sigma^{\mu\nu})^\dagger = \tfrac{1}{2}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \tag{2.3}$$

The $(\frac{1}{2}, 0)$ representation $D[\Lambda]$ satisfies the well-known relation

$$D\sigma^\mu D^\dagger = (\Lambda^{-1})^\mu_\nu \sigma^\nu \tag{2.4}$$

from which we obtain

$$\bar{D}\bar{\sigma}^\mu \bar{D}^\dagger = (\Lambda^{-1})^\mu_\nu \bar{\sigma}^\nu \tag{2.5}$$

for the $(0, \frac{1}{2})$ representation $\bar{D} = (D^\dagger)^{-1}$. Equation (2.4) is easily verified by taking an infinitesimal Λ and showing that the generators of D must be $\frac{1}{2}i\sigma^{\mu\nu}$ which satisfy (1.3) with $\mathbf{s} = \frac{1}{2}\boldsymbol{\sigma}$.

We distinguish four kinds of two-component spinor index:

$$\left.\begin{array}{ll} \text{(a)} & \phi_A \text{ denoting the transformation law } \phi \to D\phi \\ \text{(b)} & \phi^A \text{ denoting the transformation law } \phi \to \phi D^{-1} = \bar{D}^*\phi \\ \text{(c)} & \chi_{\dot{A}} \text{ denoting the transformation law } \chi \to D^*\chi = \chi\bar{D}^{-1} \\ \text{(d)} & \chi^{\dot{A}} \text{ denoting the transformation law } \chi \to \chi(D^*)^{-1} = \bar{D}\chi \end{array}\right\} \tag{2.6}$$

Because $\epsilon D = (D^\dagger)^{-1}\epsilon$ for any unimodular D, ϵ can be used as raising and lowering operator for spinor indices—(a) and (b) are equivalent, as are (c) and (d).

Now define

$$S^\mu_{A\dot{B}} = \sigma^\mu_{A\dot{B}}/\sqrt{2}, \qquad \bar{S}^{\dot{A}B}_\mu = \bar{\sigma}^{AB}_\mu/\sqrt{2} \tag{2.7}$$

$$\left.\begin{aligned} S^{\mu_1\cdots\mu_{2j}}_{A_1\cdots A_{2j}\dot{B}_1\cdots\dot{B}_{2j}} = S^{\mu_1}_{(A_1\dot{B}_1}\cdots S^{\mu_{2j}}_{A_{2j}\dot{B}_{2j})} \\ \bar{S}^{\dot{A}_1\cdots\dot{A}_{2j}B_1\cdots B_{2j}}_{\mu_1\cdots\mu_{2j}} = \bar{S}^{(\dot{A}_1B_1}_{\mu_1}\cdots\bar{S}^{\dot{A}_{2j}B_{2j})}_{\mu_{2j}} \end{aligned}\right\} \tag{2.8}$$

where the brackets denote complete symmetrisation in the two sets A and B, separately. These quantities are obviously symmetric in their tensor indices and satisfy (1.2) on account of

$$g_{\mu\nu} S^\mu_{A\dot{B}} S^\nu_{C\dot{D}} = \epsilon_{AC}\epsilon_{\dot{B}\dot{D}}, \qquad g^{\mu\nu}\bar{S}^{\dot{A}B}_\mu \bar{S}^{\dot{C}D}_\nu = \epsilon^{\dot{A}\dot{C}}\epsilon^{BD} \tag{2.9}$$

Also, from (2.4) and (2.5) it follows that, if we convert the spinor index sets to a single $(2j+1)$-fold indices m and $\dot{m}$ according to the prescription (1.7), the resulting quantities

$$S^{\mu_1\cdots\mu_{2j}}_{m\dot{n}}, \qquad \bar{S}^{\dot{n}m}_{\mu_1\cdots\mu_{2j}}$$

will satisfy (1.5) and (1.5′). Thus the quantities (2.8) are effectively the Barut–Muzinich–Williams quantities, but expressed in a notation that makes their structure readily apparent, and also makes them easier to work with.

3. *Properties of the Matrices*

Let $t_{\mu_1 \cdots \mu_{2j}}$ be an arbitrary tensor of rank $2j$ and denote its completely symmetric traceless part by

$$t_{\{\mu_1 \cdots \mu_{2j}\}}$$

The actual construction of $t_{\{\mu_1 \cdots \mu_{2j}\}}$ from $t_{\mu_1 \cdots \mu_{2j}}$ is quite complex, but will not be needed here. We define

$$\delta^{\{\nu_1 \cdots \nu_{2j}\}}_{\{\mu_1 \cdots \mu_{2j}\}} = \delta^{\nu_1}_{\{\mu_1} \cdots \delta^{\nu_{2j}}_{\mu_{2j}\}} \tag{3.1}$$

which is effectively the unit matrix for the space of completely symmetric traceless tensors. We can also define raising and lowering operators in this space, constructed from $\eta_{\mu_1 \nu_1} \cdots \eta_{\mu_{2j} \nu_{2j}}$. It will be convenient to write a completely symmetric, traceless, rank $2j$ tensor index set $\{\mu_1 \cdots \mu_{2j}\}$ as a single $(2j+1)^2$-fold index (μ). Thus (3.1) is just $\delta^{(\nu)}_{(\mu)}$, and the generalised Pauli matrices are

$$s^{(\mu)}_{\dot{n}m}, \ \bar{s}^{\dot{n}m}_{(\mu)}$$

For the generalised spinor index sets, we can define the unit matrix δ^n_m by completely symmetrising the $(2j+1)$-fold direct products of δ^B_A:

$$\delta^{(B_1}_{(A_1} \cdots \delta^{B_{2j})}_{A_{2j})}$$

and applying the prescription (1.7). Raising and lowering operators $C_{mn} = C^{mn} = C_{\dot{m}\dot{n}} = C^{\dot{m}\dot{n}}$ are obtained from $2j$-fold direct product of ϵ_{AB} also by symmetrising separately in the A's and the B's and applying (1.7). We easily obtain

$$C_{mn} = (-)^{j+m} \delta_{m,-n} \tag{3.2}$$

which identifies C_{mn} as just a Wigner 1-j symbol, as we might have expected (Brink & Satchler, 1962). The identity $\bar{\sigma}^\mu = \epsilon(\sigma^\mu)^T \epsilon^{-1}$ gives

$$\bar{s}^{(\mu)} = C(s^{(\mu)})^T C^{-1} \tag{3.3}$$

Another easily proved identity is

$$s^{(\mu)}_{m\dot{n}} \bar{s}^{\dot{p}q}_{(\mu)} = \delta^q_m \delta^{\dot{p}}_{\dot{n}} \tag{3.4}$$

which follows from the Pauli matrix identity (2.10),

$$s^\mu_{A\dot{B}} \bar{s}^{\dot{C}D}_\mu = \delta^D_A \delta^{\dot{C}}_{\dot{B}}$$

Given a spinor $\Phi^{\dot{n}m}$ belonging to the (j,j) representation we can define from it a completely symmetric traceless rank $2j$ tensor

$$\Phi^{(\mu)} = s^{(\mu)}_{m\dot{n}} \Phi^{\dot{n}m} \tag{3.5}$$

(i.e.: $\Phi^{(\mu)} = \operatorname{trace} s^{(\mu)} \Phi$ in matrix notation). Equation (3.4) shows that $\Phi^{\dot{n}m}$ can be recovered from the tensor:

$$\Phi^{\dot{n}m} = \bar{s}^{\dot{n}m}_{(\mu)} \Phi^{(\mu)} \tag{3.6}$$

This corresponds to the following two results

(a) A completely symmetric traceless rank $2j$ tensor belongs to the (j,j) representation of L.

(b) The $(2j+1)^2$ Barut–Muzinich–Williams matrices are linearly independent.

Consider the 'scalar product'

$$X^{(\mu)}\, Y_{(\mu)}$$

of two traceless symmetric tensors. From the above equations, this is just

$$X^{\dot{m}n}\, Y_{n\dot{m}}$$

where

$$X^{\dot{m}n} = \bar{s}^{\dot{m}n}_{(\mu)}\, X^{(\mu)}, \qquad Y_{n\dot{m}} = s^{(\mu)}_{n\dot{m}}\, Y_{(\mu)} \tag{3.7}$$

whence it is easy to obtain

$$\operatorname{tr} s^{(\mu)}\, \bar{s}_{(\nu)} = s^{(\mu)}_{m\dot{n}}\, s^{\dot{n}m}_{(\nu)} = \delta^{(\mu)}_{(\nu)} \tag{3.8}$$

A more complicated relation that we shall make use of is the following which contains the spin j and the spin $j+1$ quantities:

$$s^{\mu\mu_1\cdots\mu_{2j}}_{AA_1\cdots A_{2j}\dot{B}\dot{B}_1\cdots\dot{B}_{2j}}\, \bar{s}^{\dot{B}_1\cdots\dot{B}_{2j}C_1\cdots C_{2j}}_{\mu_1\cdots\mu_{2j}} = (j+1)\, \delta^{C_1}_{(A_1} \cdots \delta^{C_{2j}}_{A_{2j}} s^{\mu}_{A)\dot{B}} \tag{3.9}$$

To prove this we need the identity

$$\phi_{(A}\, \delta^{A_1}_{A_1} \cdots \delta^{A_n}_{A_n)} = N(n)\, \phi_A \tag{3.10}$$

where ϕ is any $(\frac{1}{2},0)$ spinor and N is just a number. That this must be valid is fairly obvious. We require the number N. Write

$$\begin{aligned}\phi_{(A}\, \delta^{A_1\cdots A_n}_{A_1\cdots A_n)} &= \frac{1}{n+1}\,[\phi_A\, \delta^{A_1\cdots A_n}_{(A_1\cdots A_n)} + \phi_{A_1}\, \delta^{A_1\cdots A_n}_{(A_2\cdots A_n A)} + \phi_{A_2}\, \delta^{A_1\cdots A_n}_{(A_3\cdots A_n A A_1)} + \cdots] \\ &= \frac{1}{n+1}\,[(n+1)\,\phi_A + nN(n-1)\,\phi_A]\end{aligned} \tag{3.11}$$

This gives the formula $N(n) = 1 + nN(n-1)/(n+1)$ and since $N(1) = 3/2$ we find

$$N(n) = 1 + n/2 \tag{3.12}$$

Now we have

$$\begin{aligned}s^{\mu\mu_1\cdots\mu_{2j}}_{AA_1\cdots A_{2j}\dot{B}\dot{B}_1\cdots\dot{B}_{2j}}\, \bar{s}^{\dot{B}_1\cdots\dot{B}_{2j}C_1\cdots C_{2j}}_{\mu_1\cdots\mu_{2j}} &= s^{\mu}_{(A\dot{B}}\, s^{\mu_1\cdots\mu_{2j}}_{A_1\cdots A_{2j}\dot{B}_1\cdots\dot{B}_{2j})}\, \bar{s}^{\dot{B}_1\cdots\dot{B}_{2j}C_1\cdots C_{2j}}_{\mu_1\cdots\mu_{2j}} \\ &= \delta^{C_1}_{(A_1} \cdots \delta^{C_{2j}}_{A_{2j}} s^{\mu}_{A)\,(\dot{B}}\, \delta^{\dot{B}_1}_{\dot{B}_1} \cdots \delta^{\dot{B}_{2j}}_{\dot{B}_{2j})} = N(2j)\, \delta^{C_1}_{(A_1} \cdots \delta^{C_{2j}}_{A_{2j}} s^{\mu}_{A)\dot{B}}\end{aligned}$$

which is just (3.9). Multiplying by $\bar{s}^{\dot{B}C}_{\rho}$, contracting on $\dot{B}$ and symmetrising the C's gives (substituting $2j$ for $2j+1$),

$$s^{\mu\nu_2\cdots\nu_{2j}}_{A_1\cdots A_{2j}\dot{B}_1\cdots\dot{B}_{2j}}\, \bar{s}^{\dot{B}_1\cdots\dot{B}_{2j}C_1\cdots C_{2j}}_{\rho\nu_2\cdots\nu_{2j}} = (j+\tfrac{1}{2})\,(s^{\mu}\, \bar{s}_{\rho})^{(C_1}_{(A_1}\delta^{C_2}_{A_2} \cdots \delta^{C_{2j})}_{A_{2j})}$$

which gives, since $s^{\mu}\bar{s}^{\rho} = \frac{1}{2}(g^{\mu\rho} + \sigma^{\mu\rho})$,

$$s^{\mu\nu_2\cdots\nu_{2j}}\bar{s}^{\rho}_{\nu_2\cdots\nu_{2j}} = \left(\frac{2j+1}{4}\right)(g^{\mu\rho} - (1/j)\,iM^{\mu\rho}) \tag{3.13}$$

which is the generalisation of the Pauli matrix relations (2.2) and (2.3).

4. *Zero Restmass Equations*

The equations of Dirac (1936) and Pauli & Fierz (1939) for zero restmass and spin-j are

$$\partial_{\mu}\bar{s}^{\mu\dot{B}A}\phi_{AA_2\cdots A_{2j}} = 0 \tag{4.1}$$

where ϕ is completely symmetric. That the equations (1.1) of Nelson and Good are equivalent to (4.1) is now self-evident in terms of the formalism we have set up. Multiply (4.1) by

$$\bar{s}^{\dot{B}_2\cdots\dot{B}_{2j}A_1\cdots A_{2j}}_{\mu_2\cdots\mu_{2j}}$$

and symmetrise the $\dot{B}$'s. We get

$$\partial_{\mu}\bar{s}^{\mu\dot{B}_1\cdots\dot{B}_{2j}A_1\cdots A_{2j}}_{\mu_2\cdots\mu_{2j}}\phi_{A_1\cdots A_{2j}} = 0 \tag{4.2}$$

which is just (1.1); conversely, multiplying (4.2) by

$$s^{\mu_2\cdots\mu_{2j}}_{\dot{B}_2\cdots\dot{B}_{2j}C_1\cdots C_{2j}}$$

and using (3.9) we get back the equations (4.1).

As is well known, for $j = 1$ equations (4.1) [or equivalently (1.1)] are just Maxwell's equations. Given a symmetric ϕ_{AB}, define

$$\phi_{\mu\nu} = \tfrac{1}{4}\phi_{AB}\,\sigma^{AB}_{\mu\nu}, \qquad \phi_{AB} = \tfrac{1}{2}\phi_{\mu\nu}\,\sigma^{\mu\nu}_{AB} \tag{4.3}$$

$\phi_{\mu\nu}$ is self-dual, so that written as the sum of a real and an imaginary tensor it is

$$\phi_{\mu\nu} = f_{\mu\nu} + \tfrac{1}{2}i\epsilon_{\mu\nu\rho\sigma}f^{\rho\sigma} = f_{\mu\nu} + f^{D}_{\mu\nu} \tag{4.4}$$

The complex conjugate of ϕ_{AB} is

$$\phi^{*}_{\dot{A}\dot{B}} = -\tfrac{1}{2}\phi^{*}_{\mu\nu}\,\bar{\sigma}^{\mu\nu}_{\dot{A}\dot{B}}, \qquad \phi^{*}_{\mu\nu} = f_{\mu\nu} - f^{D}_{\mu\nu} \tag{4.5}$$

The traceless symmetric tensor

$$t_{\mu\nu} = \Phi^{\dagger}\,\bar{s}_{\mu\nu}\,\Phi \tag{4.6}$$

satisfies $\partial_{\mu}t^{\mu\nu} = 0$ on account of (1.1), and as pointed out by Nelson and Good, is the energy-momentum tensor of the electromagnetic field. It is instructive, and will be useful for dealing with the $j = 2$ case, to see how this works out in our present notation. We use the identities

$$\left.\begin{aligned} \sigma^{\mu\nu}\sigma^{\rho} &= -i\epsilon^{\mu\nu\rho\lambda}\sigma_{\lambda} + g^{\rho\nu}\sigma^{\mu} - g^{\rho\mu}\sigma^{\nu} \\ \bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho} &= i\epsilon^{\mu\nu\rho\lambda}\bar{\sigma}_{\lambda} + g^{\rho\nu}\bar{\sigma}^{\mu} - g^{\rho\mu}\bar{\sigma}^{\nu} \end{aligned}\right\} \tag{4.7}$$

to obtain

$$\left.\begin{aligned}\phi_A^B\,\sigma^\rho_{B\dot C} &= 2\phi^{\mu\rho}\,\sigma_{\mu A\dot C}\\ \phi^{*\dot A}_{\dot B}\,\bar\sigma^{\rho\dot B C} &= -2\phi^{*\mu\rho}\,\bar\sigma^{\dot A C}_{\mu}\end{aligned}\right\}\tag{4.8}$$

and hence

$$\phi_A^B\,\phi^{*\dot A}_{\dot B}\,\sigma^\nu_{B\dot A}\,\bar\sigma^{\mu\dot B A} = 8\phi_\rho{}^\nu\,\phi^{*\rho\mu}\tag{4.9}$$

The tensor (4.6) can be written

$$\begin{aligned}t_{\mu\nu} &= \tfrac{1}{2}\phi^*_{\dot A\dot B}\,\bar\sigma^{\dot B A}_\mu\,\bar\sigma^{\dot A B}_\nu\,\phi_{AB}\\ &= \tfrac{1}{2}\phi^{*\dot A}_{\dot B}\,\bar\sigma^{\dot B A}\,\sigma_{\nu B\dot A}\,\phi_A^B\\ &= 4\phi_{\rho\nu}\,\phi^{*\rho}{}_\mu\\ &= 4(f_{\rho\nu}f^\rho{}_\mu - f^D_{\rho\nu}f^{D\rho}{}_\mu) + 4(f^D_{\rho\nu}f^\rho{}_\mu - f_{\rho\nu}f^{D\rho}{}_\mu)\end{aligned}$$

The two terms in brackets are respectively real and imaginary, but $t_{\mu\nu}$ in (4.6) is plainly real, so the second term must be identically zero. This also follows from the fact that it is skew in $\mu\nu$. The remaining term can be reformulated with the aid of the identity

$$\epsilon_{\rho\nu\alpha\beta}\,\epsilon^{\rho\mu\gamma\delta} = -6\delta^{\mu\gamma\delta}_{[\nu\alpha\beta]}$$

to give

$$t_{\mu\nu} = 8(f_{\rho\nu}f_{\rho\mu} - \tfrac{1}{4}g_{\mu\nu}f_{\rho\sigma}f^{\rho\sigma})\tag{4.10}$$

We are now in a position to deal with the spin-2 case. Given a completely symmetric rank 4 spinor we can define

$$\left.\begin{aligned}\phi_{\mu\nu\rho\sigma} &= \tfrac{1}{16}\phi_{ABCD}\,\sigma^{AB}_{\mu\nu}\,\sigma^{CD}_{\rho\sigma}\\ \phi_{ABCD} &= \tfrac{1}{4}\phi_{\mu\nu\rho\sigma}\,\sigma^{\mu\nu}_{AB}\,\sigma^{\rho\sigma}_{CD}\end{aligned}\right\}\tag{4.11}$$

The tensor is self-dual in each index pair $(\mu\nu)$, $(\rho\sigma)$, symmetric under interchange of the pairs, and is traceless for contraction on any two indices. Split into its real and imaginary parts it has the form

$$\phi_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \tfrac{1}{2}i\epsilon_{\rho\sigma\alpha\beta}\,R_{\mu\nu}{}^{\alpha\beta} = R_{\mu\nu\rho\sigma} + R^D_{\mu\nu\rho\sigma}\tag{4.12}$$

where the real tensor $R_{\mu\nu\rho\sigma}$ has also the symmetries of a Riemann tensor, and is traceless. The equations (1.1) in terms of this tensor are just the 'linearised Bianchi identities'

$$\partial_\mu R_{\rho\sigma\lambda\nu} + \partial_\rho R_{\sigma\mu\lambda\nu} + \partial_\sigma R_{\mu\rho\lambda\nu} = 0\tag{4.13}$$

A more detailed treatment is given in (Lord, 1971). Our present aim is to express the traceless symmetric tensor

$$t_{\mu\nu\rho\lambda} = \Phi^\dagger\,\bar s_{\mu\nu\rho\lambda}\,\Phi\tag{4.14}$$

of the $j = 2$ theory in terms of $R_{\mu\nu\rho\sigma}$. It can be rewritten

$$\begin{aligned}t_{\mu\nu\rho\lambda} &= \tfrac{1}{4}\phi^*_{\dot A\dot B\dot C\dot D}\,\bar\sigma^{\dot A A}_\mu\,\bar\sigma^{\dot B B}_\nu\,\bar\sigma^{\dot C C}_\rho\,\bar\sigma^{\dot D D}_\lambda\,\phi_{ABCD}\\ &= \tfrac{1}{4}\phi^{*\dot A\dot C}_{\ \dot B\dot D}\,\bar\sigma^{\dot B A}_\mu\,\sigma_{\nu\dot A B}\,\bar\sigma^{\dot D C}_\rho\,\sigma_{\lambda\dot C D}\,\phi^{BD}_{AC}\end{aligned}\tag{4.15}$$

where

$$\phi^*_{\dot{A}\dot{B}\dot{C}\dot{D}} = \tfrac{1}{4}\phi^*_{\mu\nu\rho\lambda}\,\bar{\sigma}^{\mu\nu}_{\dot{A}\dot{B}}\,\bar{\sigma}^{\rho\lambda}_{\dot{C}\dot{D}}$$

Using (4.9) twice the expression (4.15) is seen to be

$$\begin{aligned} t_{\mu\nu\rho\lambda} &= 4\phi_{\alpha\mu\beta\nu}\,\phi^{*\alpha}{}_{\rho}{}^{\beta}{}_{\lambda} \\ &= 4(R_{\alpha\mu\beta\nu}\,R^{\alpha}{}_{\rho}{}^{\beta}{}_{\lambda} - R^{D}_{\alpha\mu\beta\nu}\,R^{D\alpha}{}_{\rho}{}^{\beta}{}_{\lambda}) + 4(R^{D}_{\alpha\mu\beta\nu}\,R^{\alpha}{}_{\rho}{}^{\beta}{}_{\lambda} - R_{\alpha\mu\beta\nu}\,R^{D\alpha}{}_{\rho}{}^{\beta}{}_{\lambda}) \end{aligned}$$

As in the electromagnetic case, the second term must be identically zero, since it is imaginary. Hence we have

$$t_{\mu\nu\rho\lambda} = 4(R_{\alpha\mu\beta\nu}\,R^{\alpha}{}_{\rho}{}^{\beta}{}_{\lambda} - R^{D}_{\alpha\mu\beta\nu}\,R^{D\alpha}{}_{\rho}{}^{\beta}{}_{\lambda}) \tag{4.16}$$

Now a tensor of just this form has been investigated by several authors in connection with the energy-momentum of a gravitational field (Bel, 1959; Lichnerowicz, 1958; Chevreton, 1964). Bel's tensor is a rank four tensor

$$B_{\mu\nu\rho\lambda} = \tfrac{1}{2}(R_{\mu\alpha\nu\beta}\,R_{\rho}{}^{\alpha}{}_{\lambda}{}^{\beta} - {}^{D}R_{\mu\alpha\nu\beta}\,{}^{D}R_{\rho}{}^{\alpha}{}_{\lambda}{}^{\beta} - R^{D}_{\mu\alpha\nu\beta}\,R^{D}{}_{\rho}{}^{\alpha}{}_{\lambda}{}^{\beta} + {}^{D}R^{D}_{\mu\alpha\nu\beta}\,{}^{D}R^{D}{}_{\rho}{}^{\alpha}{}_{\lambda}{}^{\beta})$$

where in this case $R_{\mu\nu\rho\lambda}$ is a Riemann tensor, not necessarily traceless. In (4.17) we have used the notation

$$\begin{gathered} R^{D}_{\mu\nu\rho\lambda} = \tfrac{1}{2}i\epsilon_{\rho\lambda\alpha\beta}\,R_{\mu\nu}{}^{\alpha\beta}, \qquad {}^{D}R_{\mu\nu\rho\lambda} = \tfrac{1}{2}i\epsilon_{\mu\nu\alpha\beta}\,R^{\alpha\beta}{}_{\rho\lambda} \\ {}^{D}R^{D}_{\mu\nu\rho\lambda} = -\tfrac{1}{4}\epsilon_{\mu\nu\alpha\beta}\,\epsilon_{\rho\lambda\gamma\delta}\,R^{\alpha\beta\gamma\delta} \end{gathered} \tag{4.18}$$

Bel's tensor in general is not completely symmetric and traceless but has the partial symmetries

$$B_{\mu\nu\rho\sigma} = B_{\nu\mu\rho\sigma} = B_{\rho\sigma\mu\nu}; \qquad B_{\mu\nu\rho}{}^{\rho} = 0 \tag{4.19}$$

It is completely symmetric and traceless in the particular case when Einstein's empty space equations $R_{\mu\nu} = 0$ are satisfied. For a traceless Riemann tensor,

$$R_{\mu\nu\rho\lambda} = {}^{D}R^{D}_{\mu\nu\rho\lambda}, \tag{4.20}$$

As is easily verified by taking particular values for the indices. Thus, in the case when Einstein's equations for empty space are satisfied,

$$B_{\mu\nu\rho\lambda} = \tfrac{1}{4}t_{\mu\nu\rho\lambda} \tag{4.21}$$

with $t_{\mu\nu\rho\lambda}$ given by (4.16).

This result is highly suggestive of a connection between Einstein's theory and the equations (1.1) for $j = 2$. However, (1.1) is a flat space theory so that $R_{\mu\nu\rho\lambda}$ cannot be interpreted as a Riemann tensor. In a curved space-time the derivative of the spinor would become a covariant derivative, Fock–Ivanenko coefficients would appear in (1.1) and (4.13) would become the Bianchi identity. A rigorous treatment of the relation between Einstein's theory and the linear massless spin-2 theory is given in (Lord, 1971). An expression for Bel's tensor in the case $R_{\mu\nu} \neq 0$ that is closely related to (4.21) has been previously obtained (Lord, 1967).

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