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General relativity from gauge invariance

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Abstract. The free field equations for particles with spin are invariant under a group SL$(2, c)$ whose transformations correspond to changes of representation of the two-component spinor algebra. The generalization of the equations which extends this invariance to a guage invariance in the Yang–Mills sense necessitates the introduction of auxiliary fields (which are also necessary to maintain Lorentz covariance). These fields can be interpreted as the potentials of a spin-2 field, just as the auxiliary fields for the charge gauge group are the potentials of a spin-1 field (electromagnetism); this spin-2 field is then self-interacting. The Bargmann–Wigner formulation of the linear spin-2 field, when modified by the proposed self-interaction, yields a non-linear theory of a spin-2 field which is shown to be identical with Einstein's gravitational theory. With this interpretation the auxiliary fields take on an extra role of Yang–Mills field for the general coordinate transformation group – that is, they are the components of the affine connexion.

1. *Introduction.* Einstein's special theory of relativity (Lorentz covariance) has played a dominant part in the development of elementary particle physics. One of the most profound expressions of the requirement that physics should be consistent with special relativity is contained in the work of Bargmann and Wigner (1) where the idea that the state vector of a non-interacting particle should belong to an irreducible representation of the Poincaré group (inhomogeneous Lorentz group) is shown to lead a classification of all possible Lorentz covariant field equations for non-interacting particles; each equation is characterized by a unique spin s and restmass m. There are many formulations of these equations in the literature (1–7), all of which are equivalent. In this introduction we shall give a brief discussion of the free-field equations and set out the algebraic structures and notational conventions that we shall use in later sections.

Let x^μ $(\mu = 0, 1, 2, 3)$ be a Euclidean coordinate system in a Minkowski (flat) space-time with metric $\eta_{\mu\nu}$ $(1 = \eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33})$, and write ∂_μ for $\partial/\partial x^\mu$. The metric and its inverse $\eta^{\mu\nu}$ are used to lower and raise indices in the usual way. In order to discuss the two-component spinor representations of the homogeneous Lorentz group (8) we introduce the following 2×2 matrices:

$$\left.\begin{aligned} &\sigma^\mu = (1, \boldsymbol{\sigma}); \quad \overline{\sigma}^\mu = (1, -\boldsymbol{\sigma}),\\ &\boldsymbol{\sigma} = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}, \begin{pmatrix} & -i\\ i & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} & 1\\ -1 & \end{pmatrix}.\end{aligned}\right\} \tag{1·1}$$

The four matrices σ^μ are the basis of the Pauli algebra, or equivalently, of a two-dimensional representation of the quaternion algebra (9, 10). The notation $\bar{\sigma}^\mu$ is based on the idea of quaternion conjugation (9, 11). We find that the following relations hold:

$$\bar{\sigma}^\mu = -\epsilon\sigma^{\mu T}\epsilon, \tag{1·2}$$

$$\tfrac{1}{2}(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu) = \eta^{\mu\nu}. \tag{1·3}$$

If we define
$$\sigma^{\mu\nu} = \tfrac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu), \tag{1·4}$$

then
$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = 2(\eta^{\nu\sigma}\sigma^{\rho\mu} - \eta^{\mu\sigma}\sigma^{\rho\nu} + \eta^{\mu\rho}\sigma^{\sigma\nu} - \eta^{\nu\rho}\sigma^{\sigma\mu}), \tag{1·5}$$

which shows that $\frac{1}{2}\sigma^{\mu\nu}$ are generators of a representation of the homogeneous Lorentz group. Defining the 'dual' of a skew-symmetric tensor $f_{\mu\nu}$ by

$$f_{\overline{\mu\nu}} = \tfrac{1}{2}i\epsilon_{\mu\nu\rho\sigma}f^{\rho\sigma}, \tag{1·6}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is completely skew-symmetric with $\epsilon_{0123} = 1$, then $\sigma_{\mu\nu}$ is found to be 'self-dual'

$$\sigma_{\mu\nu} = \sigma_{\overline{\mu\nu}}. \tag{1·7}$$

Given a Lorentz transformation $\Lambda_\mu{}^\nu$ $(\Lambda_\mu{}^\rho\Lambda_\lambda{}^\sigma\eta_{\rho\sigma} = \eta_{\mu\lambda})$, its two-component spinor representation $(\frac{1}{2}, 0)$ is given by $\mathrm{SL}(2, c)$ matrix L that satisfies

$$\Lambda_\nu{}^\mu L\sigma^\nu L^\dagger = \sigma^\mu. \tag{1·8}$$

By considering the infinitesimal Lorentz matrix $\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \eta^{\nu\rho}\lambda_{\rho\mu}$ $(\lambda_{\rho\mu} = -\lambda_{\mu\rho}$ infinitesimal) we find $L = 1 + \frac{1}{4}\lambda_{\mu\nu}\sigma^{\mu\nu}$ so that the infinitesimal generators of the representation $(\frac{1}{2}, 0)$ are just $\frac{1}{2}\sigma^{\mu\nu}$. (The matrix $-L$ is also a solution but we shall not concern ourselves here with the two-to-one nature of spinor representations.) A two-component spinor (Weyl spinor) is a quantity ϕ which transforms to $L\phi$ under the Lorentz transformation $\Lambda_\mu{}^\nu$. The representation $(0, \frac{1}{2})$ is given by two-component quantities χ that transform to $(L^\dagger)^{-1}\chi$. The two representations are not equivalent. A Dirac spinor is a four-component quantity belonging to the representation $(\frac{1}{2}, 0)\oplus(\frac{1}{2}, 0)$. In the Weyl representation of the algebra of four-component spinors the decomposition into a direct sum is explicit – the 4×4 transformation matrices are in block-diagonal form so that the Dirac spinors have the form

$$\psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix}.$$

We shall write the fourfold spinor indices as capital Latin letters, thus: ψ_A.

The Weyl representation of the 'Dirac matrices' is given by

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} 1 & \\ & -1\end{pmatrix}, \tag{1·9}$$

which, by the quaternion relations already discussed, are readily seen to satisfy

$$\tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = \eta^{\mu\nu}, \tag{1·10}$$

$$\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0, \tag{1·11}$$

$$(\gamma^5)^2 = 1, \tag{1·12}$$

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = -\tfrac{1}{24}i\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \qquad (1\cdot13)$$

$$C\gamma^\mu C^{-1} = -\gamma^{\mu T} \quad \text{where} \quad C = \begin{pmatrix} \epsilon & \\ & -\epsilon \end{pmatrix}. \qquad (1\cdot14)$$

The matrix C is the 'charge conjugation matrix' in the Weyl representation. The advantage of the Weyl representation lies in the way in which it renders the relations between the two-component and four-component notation particularly transparent. Defining

$$\gamma^{\mu\nu} = \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \qquad (1\cdot15)$$

it is clear from the fact that, in the Weyl representation,

$$\gamma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & \\ & -\sigma^{\mu\nu\dagger} \end{pmatrix},$$

that $\frac{1}{2}\gamma^{\mu\nu}$ are the generators of the four-dimensional representation of the Lorentz group given by the Dirac spinors: the 4×4 spinor transformation matrix L is given in terms of the 2×2 matrix L by

$$L = \begin{pmatrix} L & \\ & (L^\dagger)^{-1} \end{pmatrix}$$

and it satisfies

$$\Lambda_\nu{}^\mu L\gamma^\nu L^{-1} = \gamma^\mu. \qquad (1\cdot16)$$

From the self-duality of $\sigma^{\mu\nu}$ we obtain the property

$$\gamma_5\gamma_{\mu\nu} = \gamma_{\mu\nu}\gamma_5 = \gamma_{\overline{\mu\nu}}. \qquad (1\cdot17)$$

We have now sufficient formalism to write down the Bargmann–Wigner formulation of the free-field equations for spin s and restmass m (which we will refer to as the BW theory of spin s and restmass m). For $s > 0$ the equations are expressed in terms of a *completely symmetric* Dirac spinor of rank $2s$ ($s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$):

$$\gamma^{\mu E}_{A_1}\partial_\mu\psi_{EA_2\ldots A_{2s}} = m\psi_{A_1\ldots A_{2s}}. \qquad (1\cdot18)$$

When m is zero there is an additional restriction

$$\gamma^{5E}_{A_1}\psi_{EA_2\ldots A_{2s}} = \psi_{A_1\ldots A_{2s}}, \qquad (1\cdot19)$$

which indicates that for massless fields the theory can be formulated in terms of a completely symmetric *Weyl* spinor of rank $2s$. For spin zero we have a special case. It is possible to express the equation for a massive spin-zero particle in the above general scheme as an equation of the same form as the spin-1 equation

$$(\gamma^{\mu E}_A\partial_\mu\psi_{EB} = m\psi_{AB})$$

but with a *skew-symmetric* instead of a symmetric spinor; the massless spin-zero equation $\Box\phi = 0$ does not fit into the scheme, but there seems to be no such field in nature anyway.

On account of the relation (1·16) the free field equations are readily seen to be invariant under the homogeneous Lorentz group:

$$\left.\begin{aligned} \psi_{A_1\ldots A_{2s}} &\to L^{B_1}_{A_1}\ldots L^{B_{2s}}_{A_{2s}}\psi_{B_1\ldots B_{2s}}, \\ \gamma^\mu &\to \gamma^\mu, \\ \partial_\mu &\to (\Lambda^{-1})_\mu{}^\nu\partial_\nu. \end{aligned}\right\} \qquad (1\cdot20)$$

Another invariance group is the group of changes of representation of the Dirac matrices:

$$\left.\begin{aligned} \psi_{A_1\ldots A_{2s}} &\rightarrow S^{B_1}_{A_1}\ldots S^{B_{2s}}_{A_{2s}}\psi_{B_1\ldots B_{2s}},\\ \gamma_\mu &\rightarrow S\gamma_\mu S^{-1},\\ \gamma_5 &\rightarrow S\gamma_5 S^{-1},\\ \partial_\mu &\rightarrow \partial_\mu, \end{aligned}\right\} \tag{1·21}$$

where S can be any 4×4 matrix, so that the invariance group is SL(4, c). We shall be concerned only with an SL(2, c) subgroup of this SL(4, c), which in section 4 we shall treat as a gauge group in the Yang–Mills sense. This is the group of change of representation of the Pauli algebra

$$\sigma^\mu \rightarrow S\sigma^\mu S^\dagger, \quad \phi \rightarrow S\phi, \quad \chi \rightarrow (S^\dagger)^{-1}\chi. \tag{1·22}$$

where S is a unimodular 2×2 matrix. The four matrices σ^μ can now no longer be considered to have the special form given by (1·1) but can be any linearly independent set of four Hermitian 2×2 matrices. The relation $\overline{\sigma}^\mu = -\epsilon\sigma^{\mu T}\epsilon$ is retained and is now considered to be the definition of $\overline{\sigma}^\mu$, which, on account of the property of ϵ that $\epsilon S = (S^T)^{-1}\epsilon$ for *any* unimodular 2×2 matrix S, have the transformation law

$$\overline{\sigma}^\mu \rightarrow (S^\dagger)^{-1}\overline{\sigma}^\mu S^{-1}. \tag{1·23}$$

All the properties that we have discussed are unaltered by this generalization. We shall continue to define the Weyl representation of γ^μ by (1·9) but where σ^μ can be any four linearly independent Hermitian matrices. The Dirac matrices in (1·18) are now supposed to be restricted to the generalized Weyl representation so that the invariance group (1·21) has a 4×4 matrix S restricted to the form

$$S = \begin{pmatrix} S & \\ & (S^\dagger)^{-1} \end{pmatrix}$$

with the 2×2 matrix S unimodular, and γ_5 is invariant. The infinitesimal generators of this 4×4 matrix S can be taken to be $\frac{1}{2}\gamma^{\mu\nu}$, but it should be borne in mind that these matrices change under the group transformations, so that it would be better to use a *fixed* representation of the Dirac algebra for these matrices, with a different kind of index.

A fundamental difficulty arising from the massive free-field equations is concerned with the introduction of a minimal electromagnetic interaction, in that the various equivalent formulations of these equations do not remain equivalent when ∂_μ is replaced throughout by the non-commuting operators $D_\mu = \partial_\mu - ieA_\mu$. In fact, if we write down the $2s$ equations obtained from (1·18) by permuting the indices on the left-hand side (making use of the symmetry of these indices on the right-hand side, or for spin zero the skew-symmetry) and replace ∂_μ by D_μ in all of them, the set of equations obtained imply

$$\left.\begin{aligned} ef_{\mu\nu}\gamma^{\mu A_1}{}_{A'}\gamma^{\nu A_2}{}_{C'}\psi_{A_1\ldots A_{2s}} &= 0,\\ f_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned}\right\} \tag{1·24}$$

Such a relationship is not consistent with a sensible physical interpretation, since it describes a geometrical relationship between the components of the electromagnetic field and those of the spin-s field, independently of the strength of the coupling. The spin-$\frac{1}{2}$ theory is free of this defect as are the zero mass theories (though zero mass fields are not coupled to the electromagnetic field in any case). This difficulty is important in the present work because we shall be dealing with another minimal interaction, with more general non-commuting operators D_μ. We shall overcome the difficulty simply by supposing that the minimal interactions are not to be applied to the massive BW theory with spin s, but to the equation obtained by forming the *sum* of the equation obtained by permuting the indices in (1·18). For spin 0 or 1 we have

$$\tfrac{1}{2}(\gamma^{\mu E}_{A} D_\mu \psi_{EB} + \gamma^{\mu E}_{B} D_\mu \psi_{AE}) = m\psi_{AB} \tag{1·25}$$

with ψ_{AB} symmetric for spin 1, skew-symmetric for spin 0. In the absence of the minimal interaction ($D_\mu \to \partial_\mu$) equation (1·25) implies (1·18) so in this case the theory described by (1·25) is equivalent to the BW theory (this circumstance follows from the identity $(\gamma^\mu \otimes 1 - 1 \otimes \gamma^\nu)(\gamma^\mu \otimes 1 + 1 \otimes \gamma^\mu)\partial_\nu \partial_\mu \equiv 0$. Equation (1·25) is in fact just the Kemmer formulation (3) of the free vector and pseudoscalar meson fields). For spin $\frac{3}{2}$ and higher, however, the theory of massive fields that we are proposing is less restrictive than the BW theory: some additional restrictions would have to be formulated. The equations formed from the sum of Bargmann–Wigner equations in the manner indicated above have been studied by Kramers, Belinfante and Lubanski (12) and by Green (13). We finally note that for zero restmass the equations

$$\left.\begin{aligned} D_\mu(\gamma^{\mu E}_{A_1}\psi_{EA_2\ldots A_{2s}} + \gamma^{\mu E}_{A_2}\psi_{A_1E\ldots A_{2s}} + \ldots \gamma^{\mu E}_{A_{2s}}\psi_{A_1\ldots E}) = 0, \\ \gamma_{5A_1}^{\;E}\psi_{EA_2\ldots A_{2s}} = \psi_{A_1\ldots A_{2s}}, \end{aligned}\right\} \tag{1·26}$$

imply the BW equation $\qquad \gamma^{\mu E}_{A_1} D_\mu \psi_{EA_2\ldots A_{2s}} = 0$

on account of the relation $\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu$ and the symmetry of $\psi_{A_1\ldots A_{2s}}$.

2. *The massless Bargmann–Wigner equations.* When $m = 0$ and $s = \frac{1}{2}$ the system of equations (1·18) and (1·19) are just the Majorana formulation of the neutrino field (14). When $m = 0$, $s = 1$ we have Maxwell's equations. To see this we note that a symmetric matrix ψ_{AB} satisfying $\gamma_{5A}^{\;E}\psi_{EB} = \psi_{EB}$ can be expressed as a linear combination of $(\gamma_{\mu\nu} + \gamma_{\overline{\mu\nu}})C^{-1}$. Thus we have

$$\psi_{AB} = \tfrac{1}{2}\psi^{\mu\nu}(\gamma_{\mu\nu}C^{-1})_{AB}, \quad \psi_{\mu\nu} = -\tfrac{1}{4}(C\gamma_{\mu\nu})^{AB}\psi_{AB}, \tag{2·1}$$

where the complex tensor $\psi^{\mu\nu}$ is self-dual. The equation $\gamma^{\mu E}_{A}\partial_\mu\psi_{EB} = 0$ expressed in terms of this tensor is just

$$\partial^\mu \psi_{\mu\nu} = 0. \tag{2·2}$$

Separating $\psi_{\mu\nu}$ into its real and imaginary parts, $\psi_{\mu\nu} = f_{\mu\nu} + f_{\overline{\mu\nu}}$, equation (2·2) gives the Maxwell equations in their familiar form

$$\partial^\mu f_{\mu\nu} = 0, \tag{2·3}$$

$$\partial_{(\mu} f_{\rho\sigma)} = \partial_\mu f_{\rho\sigma} + \partial_\rho f_{\sigma\mu} + \partial_\sigma f_{\mu\rho} = 0. \tag{2·4}$$

In the presence of charged matter only the first of these equations is modified

$$\partial^\mu f_{\mu\nu} = ej_\nu \tag{2·5}$$

but (2·4) remains the same (non-existence of magnetic monopoles), so that for either the non-interacting or the interacting electromagnetic field (2·4) implies the existence of a set of 'potentials' A_μ such that

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2·6}$$

In the spin-2 case the equations corresponding to (2·3) and (2·4) are essentially the *same* equation, so that they must both be modified in the presence of interaction and there is difficulty in defining potentials for the interacting spin-2 field, as we shall see. A completely symmetric rank 4 spinor ψ_{ABCD} satisfying $\gamma_{5A}^{\;E}\psi_{EBCD} = \psi_{ABCD}$ can be expressed in terms of a rank 4 tensor $\psi_{\mu\nu\rho\sigma}$ analogous to the spin one $\psi_{\mu\nu}$:

$$\left.\begin{aligned}\psi_{ABCD} &= \tfrac{1}{4}\psi_{\mu\nu\rho\sigma}(\gamma^{\mu\nu}C^{-1})_{AB}(\gamma^{\rho\sigma}C^{-1})_{CD},\\ \psi_{\mu\nu\rho\sigma} &= \tfrac{1}{16}\psi_{ABCD}(C\gamma_{\mu\nu})^{AB}(C\gamma_{\rho\sigma})^{CD}.\end{aligned}\right\} \tag{2·7}$$

The following properties of the tensor $\psi_{\mu\nu\rho\sigma}$ are readily apparent:

$$\psi_{\mu\nu\rho\sigma} = \psi_{\rho\sigma\mu\nu} = -\psi_{\mu\nu\sigma\rho}, \tag{2·8}$$

$$\psi_{\mu\nu\rho\sigma} = \psi_{\overline{\mu\nu}\rho\sigma} = \psi_{\mu\nu\overline{\rho\sigma}} = \psi_{\overline{\mu\nu}\,\overline{\rho\sigma}}. \tag{2·9}$$

Also, from the property (1·19) we can show that $\psi_{\mu\nu\rho\sigma}$ must be ***traceless*** in the sense that $\psi^{\mu}{}_{\nu\rho\mu} = \eta^{\mu\sigma}\psi_{\mu\nu\rho\sigma} = 0$. On account of the skew-symmetry of C, the contraction of the second equation (2·7) can be written as

$$\psi^{\mu}{}_{\nu\rho\mu} = \tfrac{1}{16}\psi_{ABCD}(C\gamma_\nu\gamma^\mu)^{AB}(C\gamma_\mu\gamma_\rho)^{CD} \tag{2·10}$$

and on account of the symmetry of ψ_{ABCD} and (1·19), ψ_{ABCD} can be written in the form

$$\psi_{ABCD} = \tfrac{1}{2}\psi^{\lambda\sigma}_{AD}(\gamma_{\lambda\sigma}C^{-1})_{BC}, \tag{2·11}$$

which, when substituted into (2·10) gives

$$\psi^{\mu}{}_{\nu\rho\mu} = \tfrac{1}{32}\psi^{\lambda\sigma}_{AD}[C\gamma_\nu\gamma^\mu\gamma_{\lambda\sigma}\gamma_\mu\gamma_\rho]^{AD} \tag{2·12}$$

which vanishes on account of the identity $\gamma^\mu\gamma_{\lambda\sigma}\gamma_\mu = 0$. Thus

$$\psi^{\mu}{}_{\nu\rho\mu} = 0. \tag{2·13}$$

By virtue of (2·9) this traceless property can be re-expressed as a cyclic symmetry

$$\psi_{(\mu\nu\rho)\sigma} = \psi_{\mu\nu\rho\sigma} + \psi_{\nu\rho\mu\sigma} + \psi_{\rho\mu\nu\sigma} = 0, \tag{2·14}$$

which is in fact only *one* restriction of the tensor components

$$\psi_{0123} + \psi_{0231} + \psi_{0312} = 0 \tag{2·15}$$

since if any two of the indices in (2·14) are equal it reduces to one of the properties (2·8). That (2·8), (2·9) and (2·14) are *all* the symmetries of $\psi_{\mu\nu\rho\sigma}$ that follow from the symmetry of ψ_{ABCD} and the property (1·19) can be seen by noting that since a self dual rank 2 tensor has only 3 components, the number of components of a rank 4 tensor satisfying (2·8) and (2·9) is the same as that of a symmetric 3×3 matrix, i.e. 6 components. The restriction (2·15) then reduces the number of linearly independent components to 5, which corresponds to the number of components of ψ_{ABCD} which is equivalent to a completely symmetric Weyl spinor of rank 4 (a completely symmetric quantity of rank 4 in *two* dimensions – see the remark following (1·19)).

The BW equation
$$\gamma^{\mu E}_{A}\partial_\mu \psi_{EBCD} = 0 \tag{2·16}$$
can now be re-expressed in terms of the tensor. It gives just
$$\partial^{\mu}\psi_{\mu\nu\rho\sigma} = 0. \tag{2·17}$$
We now introduce a real tensor $A_{\mu\nu\rho\sigma}$ which corresponds to the $f_{\mu\nu}$ in the spin-1 case. We note that any complex tensor with the symmetries (2·9) and (2·10) can be expressed in terms of a real tensor $A_{\mu\nu\rho\sigma}$ according to
$$\psi_{\mu\nu\rho\sigma} = \tfrac{1}{2}(A_{\mu\nu\rho\sigma}+A_{\overline{\mu\nu}\,\overline{\rho\sigma}})+\tfrac{1}{2}(A_{\overline{\mu\nu}\,\rho\sigma}+A_{\mu\nu\,\overline{\rho\sigma}}) \tag{2·18}$$
where
$$A_{\mu\nu\,\rho\sigma} = A_{\rho\sigma\,\mu\nu} = -A_{\mu\nu\,\sigma\rho}. \tag{2·19}$$
The traceless property of $\psi_{\mu\nu\rho\sigma}$ implies a similar property $A_{\mu\nu\rho\sigma}$ and of $A_{\overline{\mu\nu}\,\rho\sigma}$, so that $A_{\mu\nu\rho\sigma}$ satisfies
$$A^{\mu}{}_{\nu\rho\mu} = 0, \tag{2·20}$$
$$A_{(\mu\nu\rho)\sigma} = 0. \tag{2·21}$$
But now (2·19), (2·20) and (2·21) imply
$$A_{\overline{\mu\nu}\,\overline{\rho\sigma}} = A_{\mu\nu\,\rho\sigma}; \tag{2·22}$$
for instance,
$$A_{\overline{12}\,12} = -A_{30\,30} = -A_{31\,31}-A_{32\,32} \text{ (from the traceless property).}$$
also
$$-A_{30\,30} = A_{10\,10}+A_{20\,20};$$
$$A_{12\,12} = -A_{13\,13}+A_{10\,10}$$
$$= -A_{32}A_{32}+A_{02\,02}.$$
Hence $A_{\overline{12}\,\overline{12}} = \frac{1}{2}(A_{10\,10}+A_{20\,20}-A_{31\,31}-A_{32\,32}) = A_{12\,12}$. The other components of (2·22) can be proved in a similar manner. We can now write the complex tensor $\psi_{\mu\nu\rho\sigma}$ more simply as
$$\psi_{\mu\nu\rho\sigma} = A_{\mu\nu\rho\sigma}+A_{\overline{\mu\nu}\,\rho\sigma}. \tag{2·23}$$
The number of components of a tensor satisfying (2·19) is 21 (symmetric 6×6 matrix). The traceless property (2·20) is a set of 10 restrictions and the cyclic symmetry (2·21) is one restriction, so that $A_{\mu\nu\rho\sigma}$ has 10 linearly independent components which correspond to the 5 complex components of $\psi_{\mu\nu\rho\sigma}$. The set of restrictions on $A_{\mu\nu\rho\sigma}$ must therefore be complete. Expressing the spin-2 equation (2·17) now in terms of the real tensor we have
$$\partial^{\mu}A_{\mu\nu\rho\sigma} = 0, \tag{2·24}$$
$$\partial^{\mu}A_{\overline{\mu\nu}\,\rho\sigma} = 0, \tag{2·25}$$
which are entirely analogous to the spin-1 equations (2·3) and (2·4), except that, due to (2·22), the equations (2·24), (2·25) are not distinct but simply two ways of writing the same equation.

We now wish to introduce potentials for the BW spin-2 field, the analogues of equation (2·6). Writing (2·25) as
$$\partial_{(\mu}A_{\nu\rho)\sigma\lambda} = 0$$

we see that $A_{\nu\rho\sigma\lambda}$ can be obtained from a tensor $A_{\rho\sigma\lambda} = A_{\rho\lambda\sigma}$:

$$A_{\nu\rho\sigma\lambda} = \partial_\nu A_{\rho\sigma\lambda} - \partial_\rho A_{\nu\sigma\lambda}, \tag{2·26}$$

and we investigate the properties of $A_{\rho\sigma\lambda}$ implied by the symmetries of $A_{\nu\rho\sigma\lambda}$. The cyclic symmetry $A_{(\nu\rho\sigma)\lambda} = 0$ leads to

$$\partial_\nu A_{[\rho\sigma]\lambda} + \partial_\rho A_{[\sigma\nu]\lambda} + \partial_\sigma A_{[\nu\rho]\lambda} = 0, \tag{2·27}$$

where
$$A_{[\rho\sigma]\lambda} = A_{\rho\sigma\lambda} - A_{\sigma\rho\lambda}.$$

Equation (2·27) implies that $A_{[\rho\sigma]\lambda}$ can be obtained from a set of 'secondary potentials' $A_{\sigma\lambda}$,

$$A_{[\rho\sigma]\lambda} = \partial_\rho A_{\sigma\lambda} - \partial_\sigma A_{\rho\lambda}. \tag{2·28}$$

We now note that the potentials $A_{\rho\sigma\lambda}$ are not uniquely specified by (2·26); the right-hand side is unchanged if we add to $A_{\rho\sigma\lambda}$ any quantity of the form $\partial_\rho a_{\sigma\lambda}$ where $a_{\sigma\lambda}$ is skew-symmetric. (This is to be compared to the electromagnetic case, where (2·5) is unchanged if we add an arbitrary gradient $\partial_\mu a$ to A_μ.) If we choose $a_{\sigma\lambda}$ to be the skew part of $\frac{1}{2}(A_{\lambda\sigma} - A_{\sigma\lambda})$ then the new $A_{[\rho\sigma]\lambda}$ in (2·28) is expressed only in terms of the symmetric part of $A_{\sigma\lambda}$. Thus we are at liberty to 'fix the gauge' so that, in (2·28),

$$A_{\sigma\lambda} = A_{\lambda\sigma}. \tag{2·29}$$

In this case, the identity

$$A_{\mu\rho\sigma} = \tfrac{1}{2}(A_{[\sigma\mu]\rho} + A_{[\mu\rho]\sigma} - A_{[\rho\sigma]\lambda}) \tag{2·30}$$

gives
$$A_{\mu\rho\sigma} = A_{[\sigma\rho]\mu} \tag{2·31}$$

and hence
$$A_{\mu\rho\sigma} = \partial_\sigma A_{\rho\mu} - \partial_\rho A_{\sigma\mu}, \tag{2·32}$$

$$A_{(\mu\rho\sigma)} = 0, \tag{2·33}$$

$$\partial_\lambda A_{\mu\rho\sigma} + \partial_\rho A_{\mu\sigma\lambda} + \partial_\sigma A_{\mu\lambda\rho} = 0. \tag{2·34}$$

It will be convenient to define the field

$$\left.\begin{aligned} B_{\mu\sigma\rho} &= \tfrac{1}{2}(A_{\mu\rho\sigma} + \partial_\mu A_{\rho\sigma}) \\ &= \tfrac{1}{2}(\partial_\sigma A_{\rho\mu} + \partial_\mu A_{\rho\sigma} - \partial_\rho A_{\mu\sigma}) \end{aligned}\right\} \tag{2·35}$$

which in symmetric in $(\mu\sigma)$. We then have

$$A_{\mu\nu\rho\sigma} = 2(\partial_\nu B_{\mu\sigma\rho} - \partial_\mu B_{\nu\sigma\rho}). \tag{2·36}$$

We now consider the equations that the secondary potentials must satisfy in order that $A_{\mu\nu\rho\sigma}$ be traceless. These are easily seen to be

$$\Box A_{\mu\nu} - \partial^\rho \partial_\nu A_{\mu\rho} - \partial^\rho \partial_\mu A_{\nu\rho} + \partial_\mu \partial_\nu A = 0, \tag{2·37}$$

where $A = A^\mu_\mu = \eta^{\mu\nu} A_{\mu\nu}$. These equations are precisely the *linearized* version of Einstein's gravitational equations (15). However, the conception is quite different in that here we have an *essentially linear* spin-2 theory for which (2·37) are exact equations – they are in no sense a 'weak field approximation'. Also the space-time is flat; the metric is $\eta_{\mu\nu}$. In Einstein's treatment we would have a curved space-time with metric $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon A_{\mu\nu}$ where ϵ is some infinitesimal parameter. Equation (2·37) in Einstein's theory has extra terms of order ϵ. The Christoffel symbols $\{^{\rho}_{\mu\nu}\}$ formed from $g_{\mu\nu}$ are equal to $\epsilon B_{\mu\nu}{}^{\rho}+O(\epsilon^2)$ and the Riemann tensor would be $\frac{1}{2}\epsilon A_{\mu\nu\rho\sigma}+O(\epsilon^2)$; it is the Riemann tensor, rather than $A_{\mu\nu\rho\sigma}$ that is supposed to be *exactly* traceless. Both the Reimann tensor and $A_{\mu\nu\rho\sigma}$ satisfy the cyclic symmetry exactly from the way they have been constructed, and (2·25) corresponds to the Bianchi identity of the Riemann tensor.

3. *Interaction of the spin-2 field.* We consider now, in a completely general way the modification of the BW spin-2 equation in the presence of interactions. We insert a 'current' in the right-hand side of (2·16) (equivalently (2·24) or (2·25))

$$\partial^{\mu}A_{\mu\nu\rho\sigma} = fj_{\nu\rho\sigma}, \tag{3·1}$$

where $j_{\nu\rho\sigma}$ is constructed in some way from the other fields and possibly (in the case of self interaction) from the spin-2 field itself, and f is a coupling constant analogous to e in the spin-1 theory. The tensor $A_{\mu\nu\rho\sigma}$ of course still has the symmetries (2·19–22), which imply the following properties of $j_{\nu\rho\sigma}$:

$$j_{\nu\rho\sigma} = -j_{\nu\sigma\rho}, \tag{3·2}$$

$$j^{\mu}{}_{\mu\sigma} = 0, \tag{3·3}$$

$$j_{(\mu\rho\sigma)} = 0, \tag{3·4}$$

$$\partial^{\mu}j_{\mu\rho\sigma} = 0. \tag{3·5}$$

Now, since (2·25) is no longer valid, we cannot define potentials according to (2·26). Instead, we will solve the problem of potentials in the interacting spin-2 field by constructing a tensor $B_{\mu\nu\rho\sigma}$ from $A_{\mu\nu\rho\sigma}$ and $j_{\nu\rho\sigma}$, which will satisfy $\partial_{(\lambda}B_{\mu\nu)\rho\sigma}=0$ and hence will have the form $\partial_{\mu}A_{\nu\rho\sigma}-\partial_{\nu}A_{\mu\rho\sigma}$. In order to do this we will make an *assumption* about the form of the current $j_{\nu\rho\sigma}$. We will assume the interaction to be such that

$$\partial_{\mu}j_{\nu\rho\sigma}+\partial_{\rho}j_{\nu\sigma\mu}+\partial_{\sigma}j_{\nu\mu\rho} = 0. \tag{3·6}$$

If this is valid, then $j_{\nu\rho\sigma}$ has the form

$$j_{\nu\rho\sigma} = \partial_{\rho}j_{\sigma\nu}-\partial_{\sigma}j_{\rho\nu} \tag{3·7}$$

for some tensor $j_{\sigma\nu}$. The cyclic symmetry (3·4) then implies the existence of a vector j_{ν} such that

$$j_{[\mu\nu]} = \partial_{\mu}j_{\nu}-\partial_{\nu}j_{\mu}. \tag{3·8}$$

We have some freedom of choice for $j_{\sigma\nu}$ in that $j_{\nu\rho\sigma}$ given by (3·7) is unaltered by adding $\partial_{\sigma}a_{\nu}$ to $j_{\sigma\nu}$, with a_{ν} an arbitrary vector. Choosing $a_{\mu}=-j_{\mu}$ we can transform away $j_{[\mu\nu]}$ by making use of this freedom of choice. Hence (3·7) possesses a *symmetric* solution

$$j_{\mu\nu} = j_{\nu\mu}. \tag{3·9}$$

We now define the tensor

$$B_{\mu\nu\rho\sigma} = A_{\mu\nu\rho\sigma}+f(\eta_{\mu\rho}j_{\nu\sigma}-\eta_{\nu\rho}j_{\mu\sigma}+\eta_{\nu\sigma}j_{\mu\rho}-\eta_{\mu\sigma}j_{\nu\rho}) \tag{3·10}$$

which has the following symmetries in common with $A_{\mu\nu\rho\sigma}$:

$$B_{\mu\nu\rho\sigma} = B_{\rho\sigma\mu\nu} = -B_{\mu\nu\sigma\rho}, \tag{3·11}$$

$$B_{\mu(\nu\rho\sigma)} = 0. \tag{3·12}$$

However, unlike $A_{\mu\nu\rho\sigma}$, it is *not* traceless:

$$B_{\mu\nu} = \eta^{\rho\sigma} B_{\rho\mu\nu\sigma} = -f(2j_{\mu\nu} + \eta_{\mu\nu} j), \tag{3·13}$$

where $j = j^{\mu}_{\mu} = \eta^{\mu\nu} j_{\mu\nu}$.

In terms of this new tensor $B_{\mu\nu\rho\sigma}$, the basic equation (3·1) can be written

$$\partial^{\mu} B_{\mu\nu\rho\sigma} = f[\partial_{\rho}(2j_{\sigma\nu} + \eta_{\sigma\nu} j) - \partial_{\sigma}(2j_{\rho\nu} + \eta_{\rho\nu} j)]. \tag{3·14}$$

(We have made use of $\partial^{\mu} j_{\mu\rho} = \partial_{\mu} j$, which follows from $j^{\mu}{}_{\mu\rho} = 0$.) We now show that $\partial_{(\mu} B_{\nu\rho)\sigma\lambda} = 0$ as follows:

$$B_{\overline{\mu\nu}\,\rho\sigma} = A_{\overline{\mu\nu}\,\rho\sigma} + fi\epsilon_{\mu\nu\alpha\lambda}(\delta^{\alpha}_{\rho} j^{\lambda}_{\sigma} - \delta^{\alpha}_{\sigma} j^{\lambda}_{\rho})$$

so that

$$\partial^{\mu} B_{\overline{\mu\nu}\,\rho\sigma} = fj_{\nu\,\overline{\rho\sigma}} + fi\epsilon_{\mu\nu\alpha\lambda}(\delta^{\alpha}_{\rho}\partial^{\mu} j^{\lambda}_{\sigma} - \delta^{\alpha}_{\sigma}\partial^{\mu} j^{\lambda}_{\rho}).$$

Therefore,

$$\partial^{\mu} B_{\overline{\mu\nu}\,\overline{\rho\sigma}} = fj_{\nu\rho\sigma} - f\epsilon_{\rho\sigma\gamma\delta}\epsilon_{\mu\nu\alpha\lambda}(\eta^{\alpha\gamma}\partial^{\mu} j^{\lambda\delta})$$

$$= f(j_{\nu\rho\sigma} - \partial_{\rho} j_{\sigma\nu} + \partial_{\sigma} j_{\rho\nu})$$

(where we have used $\epsilon_{\rho\sigma\delta\gamma}\epsilon^{\mu\nu\lambda\gamma} = -\delta^{\mu}_{(\rho}\delta^{\nu}_{\sigma}\delta^{\lambda}_{\delta)} + \delta^{\mu}_{(\sigma}\delta^{\nu}_{\rho}\delta^{\lambda}_{\delta)}$). We have therefore shown that $\partial^{\mu} B_{\overline{\mu\nu}\,\overline{\rho\sigma}} = 0$, or equivalently $\partial^{\mu} B_{\overline{\mu\nu}\,\rho\sigma} = 0$, that is,

$$\partial_{(\mu} B_{\lambda\nu)\rho\sigma} = 0. \tag{3·15}$$

The potentials can now be defined in terms of $B_{\lambda\nu\rho\sigma}$:

$$B_{\lambda\nu\rho\sigma} = \partial_{\lambda} A_{\nu\rho\sigma} - \partial_{\nu} A_{\lambda\rho\sigma}. \tag{3·16}$$

We can make use of (3·11, 12) to define symmetric secondary potentials as before,

$$A_{\mu\,\rho\sigma} = \partial_{\sigma} A_{\rho\mu} - \partial_{\rho} A_{\sigma\mu}, \tag{3·17}$$

and the symmetry properties (2·32–34) remain valid. The traceless property of the rank 4 tensor is now replaced by (3·13) so that (2·37) becomes

$$\Box A_{\mu\nu} - \partial^{\rho}\partial_{\nu} A_{\mu\rho} - \partial^{\rho}\partial_{\mu} A_{\nu\rho} + \partial_{\mu}\partial_{\nu} A = -f[2j_{\mu\nu} + \eta_{\mu\nu} j]. \tag{3·18}$$

The similarity of form between the linearized Einstein theory and the BW theory described here, that we discussed briefly at the end of the previous section, can now be extended to the case of interaction with matter. The Ricci tensor in Einstein theory will now be $\frac{1}{2}\epsilon B_{\mu\nu} + O(\epsilon^2)$ so that the Einstein tensor is $\frac{1}{2}\epsilon(B_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} B) + O(\epsilon^2)$ where $B = \eta^{\mu\nu} B_{\mu\nu}$. Thus the Einstein equations are

$$\tfrac{1}{2}\epsilon(B_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} B) + O(\epsilon^2) = \kappa T_{\mu\nu},$$

where $T_{\mu\nu}$ is the stress-energy tensor of the matter field, which is $O(\epsilon)$ for weak fields. Thus, we see that the similarity of form of the two theories persists in the interacting case if we set $\eta_{\mu\nu} j - j_{\mu\nu} = (\kappa/f\epsilon) T_{\mu\nu} + O(\epsilon)$. In this connexion it is interesting to recall that $\partial^{\mu}(\eta_{\mu\nu} j - j_{\mu\nu}) = 0$. (It is to be emphasized that the formal similarities with linearized gravitational theory are pointed out here only for their intrinsic interest, and the discussion is not meant to imply any *interpretation* of the BW spin-2 theory, which is essentially a theory in flat space-time.)

Finally, we write down an expression for $A_{\mu\nu\rho\sigma}$ in terms of $B_{\mu\nu\rho\sigma}$, which is essentially an expression of $A_{\mu\nu\rho\sigma}$ in terms of the potentials without reference to the currents. We have

$$B_{\mu\nu} = -(2j_{\mu\nu} + \eta_{\mu\nu} j) \tag{3·19}$$

and hence
$$B = -6j, \tag{3·20}$$

so that
$$j_{\mu\nu} = \frac{1}{2}\left(\eta_{\mu\nu}\frac{B}{6} - B_{\mu\nu}\right); \tag{3·21}$$

substituting this expression in (3·10) then gives

$$A_{\mu\nu\rho\sigma} = B_{\mu\nu\rho\sigma} + \tfrac{1}{2}(\eta_{\mu\rho}B_{\nu\sigma} - \eta_{\mu\sigma}B_{\nu\rho} + \eta_{\nu\sigma}B_{\mu\rho} - \eta_{\nu\rho}B_{\mu\sigma}) - \tfrac{1}{6}B(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}). \tag{3·22}$$

4. *Change of representation as a gauge group.* We come now to the central idea behind this paper, which is the generalization of the transformation matrix S of (1·22) to a gauge transformation in the Yang–Mills sense (16) – we will let S vary over space-time. The SL(2, c) subgroup of the SL(4, c) transformations (1·21) now has 4×4 matrices $S(x)$ of the form
$$S(x) = \begin{pmatrix} S(x) & \\ & [S^\dagger(x)]^{-1} \end{pmatrix}, \tag{4·1}$$

where the 2×2 matrix $S(x)$ is unimodular, and space-time dependent. The Pauli matrices in (1·22) must of course also be space-time dependent: define $\sigma^a(x)$ ($a = 0, 1, 2, 3$) to be an arbitrary set of linearly independent Hermitian 2×2 matrix fields, subject to the transformation
$$\sigma^a(x) \to S(x)\,\sigma^a(x)\,S^\dagger(x) \tag{4·2}$$

and define
$$\overline{\sigma}^a(x) = -\epsilon[\sigma^a(x)]^T\epsilon, \quad \epsilon = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \tag{4·3}$$

which then transforms to
$$[S^\dagger(x)]^{-1}\,\overline{\sigma}^a(x)\,S^{-1}(x) \tag{4·4}$$

under (4·2). The matrices
$$\gamma^a(x) = \begin{pmatrix} & \sigma^a(x) \\ \overline{\sigma}^a(x) & \end{pmatrix} \tag{4·5}$$

transform to $S(x)\,\gamma^a(x)\,S^{-1}(x)$ under (4·2), with $S(x)$ given by (4·1). (Note that we are now using small Latin indices $a, b\ldots$ for vector component labels rather than $\mu, \nu, \ldots$ of previous sections. The reason for this change of notation will become apparent but at the moment no significance is attached to it.) The operator ∂_μ in the free-field equations occurs in the combination $\gamma^\mu\partial_\mu$. We shall replace this operator by $\gamma^a(x)\,d_a$ where the d_a are differential operators satisfying
$$\left.\begin{aligned} d_a(AB) &= (d_aA)\,B + A d_a B, \\ d_a C &= 0 \text{ for constant } C; \end{aligned}\right\} \tag{4·6}$$

we will *not* assume they commute, but instead we write
$$[d_a, d_b] = X_{ab}{}^c(x)\,d_c. \tag{4·7}$$

Note that this is *not* an added assumption – at this stage we could set the $X_{ab}{}^c$ zero and identify the d_a with $\partial/\partial x^a$ where x^a is a Euclidean coordinate system in flat space-time. Even with non-zero $X_{ab}{}^c$ we can consider the d_a to be *linear combinations* of the $\partial/\partial x^a$ with real coefficients (which satisfy a relation of the form (4·7)) and then $\gamma^a(x)\,d_a(x) = \gamma'^a(x)\,\partial/\partial x^a$ where $\gamma'^a(x)$ is obtained from $\gamma^a(x)$ by replacing $\sigma^a(x)$ by a linear combination of themselves, which of course makes no difference since the $\sigma^a(x)$ are arbitrary. However, the assumption that the d_a are linear combinations of

$\partial/\partial x^a$ for some Euclidean coordinates x^a will not be made. We shall use only the properties (4·6, 7) of the d_a. The Jacobi identity $d_{(abc)} = 0$ where $d_{abc} = [d_a[d_b d_c]]$ gives the following equation that the $X_{ab}{}^c$ must satisfy

$$d_{(a} X_{bc)}{}^f + X_{(bc}{}^e X_{a)e}{}^f = 0. \tag{4·7a}$$

The Lorentz transformation for a Dirac spinor is still given by the matrix L of equation (1·8), but with $\sigma^a(x)$ dependent on space-time the matrix L will be also (though of course Λ^b_a are still constants),

$$\sigma^a(x) = \Lambda^a_b L(x)\,\sigma^b(x)\,L^\dagger(x). \tag{4·8}$$

The infinitesimal generators of $L(x)$ are then

$$\tfrac{1}{2}\sigma^{ab}(x) = \tfrac{1}{4}(\sigma^a(x)\,\overline{\sigma}^b(x) - \sigma^b(x)\,\overline{\sigma}^a(x)) \tag{4·9}$$

so that $L(x)$ is space-time dependent through its generators – the parameters of the Lorentz group are of course constants. The Lorentz group itself then becomes in a sense a Yang–Mills group, but in rather a different way from the usual kind of Yang–Mills group where it is the parameters that are space-time dependent. For a treatment of the Lorentz group as a Yang–Mills group in the usual sense see Utiyama(17) and Kibble(18). These works make use of a Lagrangian formalism (which we are avoiding here) and by a suitable simple choice of Lagrangian for their auxiliary fields these fields can be identified as Einstein's gravitational fields. Our use of Bargmann–Wigner formalism rather than Lagrangians (the BW equations are particularly complicated to deal with in Lagrangian theory (19)) enables us to relate gravitation to the BW spin-2 theory. In any case our Lorentz group is at this stage not really a Yang–Mills group since the Λ^a_b in (4·8) are constants.

Defining the 4×4 matrix

$$L(x) = \begin{pmatrix} L(x) & \\ & [L^\dagger(x)]^{-1} \end{pmatrix}, \tag{4·10}$$

the equations

$$\gamma^{aE}{}_{A_1}(x)\, d_a \psi_{EA_2\ldots A_{2s}} = m\psi_{A_1\ldots A_{2s}} \tag{4·11}$$

are *not* invariant under the Lorentz rotation

$$\left.\begin{aligned} \psi_{A_1\ldots A_{2s}} &\to L^B_{A_1}\ldots L^{B_{2s}}_{A_{2s}}\psi_{B_1\ldots B_{2s}},\\ \gamma^a &\to \gamma^a \quad (= \Lambda^a_b L\gamma^a L^{-1}),\\ d_a &\to (\Lambda^{-1})^b_a\, d_b, \end{aligned}\right\} \tag{4·12}$$

due to the non-vanishing of $d_a L$ (for notational convenience we will drop the convention of indicating the x-dependence of the various quantities explicitly). We must replace d_a in (4·11) by D_a where

$$D_a\psi_{A_1\ldots A_{2s}} = d_a\psi_{A_1\ldots A_{2s}} - \Gamma^{\ E}_{aA_1}\psi_{EA_2A_{2s}} - \Gamma^{\ E}_{aA_2}\psi_{A_1E\ldots A_{2s}}\ldots - \Gamma^{\ E}_{aA_{2s}}\psi_{A_1\ldots E} \tag{4·13}$$

where Γ_a is a set of auxiliary fields with the Lorentz transformation law

$$\Gamma_a \to (\Lambda^{-1})^b_a L\Gamma_a L^{-1} + L d_a L^{-1}. \tag{4·14}$$

(The definition (4·13) is extended to quantities with upper spinor indices by the requirement that for a Lorentz scalar D_a is equal to d_a. Then

$$D_a(\overline{\psi}^A\psi_A) = d_a(\overline{\psi}^A\psi_A) \quad \text{gives} \quad D_a\overline{\psi}^A = d_a\overline{\psi}^A + \overline{\psi}^B\Gamma^A_B.)$$

The matrices Γ_a can be taken to be a linear combination of $\gamma_{ab} = \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)$:

$$\Gamma_a = \tfrac{1}{2}g a_{abc}\gamma^{bc} \tag{4·15}$$

where g is a universal coupling constant for the minimal coupling of a_{abc} to all BW fields. The transformation law of the a_{abc} is

$$a_{abc} \to (\Lambda^{-1})^e_a(\Lambda^{-1})^f_b a_{efg} - \frac{1}{g}\,\mathrm{tr}\,[\gamma_{bc}Ld_aL^{-1}]. \tag{4·16}$$

The need for a generalized derivative (4·13) exists only for spinors, since for vectors and tensors the Lorentz transformation matrices are constants. However, it will be convenient to define, for any vector v_a, the quantity

$$D_a v_b = d_a v_b - C_{ba}{}^e v_e, \tag{4·17}$$

and hence

$$D_a v^b = d_a v^b + C_{ea}{}^b v^e, \tag{4·18}$$

where $C_{ea}{}^b$ has a tensor transformation law (i.e. its transformation law has no extra terms like the final term in (4·16)) and will be chosen so that

$$D_a\gamma_b = d_a\gamma_b - [\Gamma_a, \gamma_b] - C_{ba}{}^e\gamma_e = 0. \tag{4·19}$$

That this is possible is seen by noting that $d_a\gamma_b$ is a linear combination of the γ_e due to (4·6) and $[\Gamma_a, \gamma_b]$ is a linear combination of the γ_e due to (4·15). We obtain

$$C_{ba}{}^e = g a_{ab}{}^e + \tfrac{1}{2}\,\mathrm{tr}\,[(d_a\gamma_b)\gamma^e]. \tag{4·20}$$

Under a Lorentz transformation the second term in this expression becomes

$$\tfrac{1}{2}(\Lambda^{-1})^f_a\,\mathrm{tr}\,[(d_f\gamma_b)\gamma^e] \tag{4·21}$$

which, by making use of

$$\gamma^a = \Lambda^a_b L\gamma^b L^{-1}, \tag{4·22}$$

can be shown to be equal to

$$(\Lambda^{-1})^f_a(\Lambda^{-1})^g_b\Lambda^e_h\,\tfrac{1}{2}\,\mathrm{tr}\,[(d_f\gamma_g)\gamma^h] + \tfrac{1}{2}\,\mathrm{tr}\,[\gamma_b{}^e Ld_aL^{-1}]. \tag{4·23}$$

The first term in (4·23) is just the tensor part of the transformation law of

$$\tfrac{1}{2}\,\mathrm{tr}\,[(d_a\gamma_b)\gamma^e],$$

while the second term exactly cancels the extra term in the transformation of $ga_{ab}{}^e$. Hence $C_{ba}{}^e$ is just a tensor under Lorentz transformations.

The tensor C_{bae} is skew-symmetric in (be) since $(d_a\gamma_b)\gamma_e = -\gamma_b d_a\gamma_e$ by the constancy of $\eta_{be} = \frac{1}{2}(\gamma_b\gamma_e + \gamma_e\gamma_b)$:

$$C_{bae} = -C_{eab}. \tag{4·24}$$

The generalized derivative of the constant tensor ϵ_{abcd} is then

$$\begin{aligned} D_e\epsilon_{abcd} &= C_{ae}{}^f\epsilon_{fbcd} + C_{be}{}^f\epsilon_{afcd} + C_{ce}{}^f\epsilon_{abfd} + C_{de}{}^f\epsilon_{abcf} \\ &= C_{fe}{}^f\epsilon_{abcd} = 0 \end{aligned} \tag{4·25}$$

(the second line is proved from the first by assigning specific values to $abcd$).

With (4·19) and (4·25) we see that in the generalized BW equations

$$\gamma^{aE}{}_{A_1} D_a\psi_{EA_2\ldots A_{2s}} = m\psi_{A_1\ldots A_{2s}}$$

we can pass to the tensor formulation simply by replacing ∂_a in the tensor formulations of section 2 by replacing ∂_a by D_a. Thus for spin 2

$$\gamma^{aE}_{\ A} D_a \psi_{EBCD} = 0 \tag{4·26}$$

implies

$$D^a A_{abcd} = 0, \tag{4·27}$$

or equivalently,

$$D_{(a} A_{bc)de} = 0. \tag{4·28}$$

We now see that we have again the difficulty of introducing potentials, since with (4·28) in place of (2·25) we can no longer write A_{abcd} in a form analogous to (2·26). We have a similar difficulty in the spin-1 case, where (2·4) is replaced by

$$D_{(a} f_{bc)} = 0. \tag{4·29}$$

These difficulties will be solved in the next section by identifying the auxilliary fields a_{abc} as the *potentials of the spin-2 field*.

We now note how the equations generalized by replacing d_a by D_a are already invariant under the gauge transformations

$$\left.\begin{aligned} \psi_{A_1 A_{2s}} &\to S^{B_1}_{A_1} \ldots S^{B_{2s}}_{A_{2s}} \psi_{B_1 \ldots B_{2s}}, \\ \gamma^a &\to S\gamma^a S^{-1}, \\ \gamma^5 &\to \gamma^5, \\ d_a &\to d_a, \end{aligned}\right\} \tag{4·30}$$

where

$$S = \begin{pmatrix} S(x) & \\ & [S^\dagger(x)]^{-1} \end{pmatrix}$$

for any unimodular space-time dependent 2×2 matrix $S(x)$, provided the Γ_a are identified as the auxiliary fields for the Yang–Mills group (4·30), with transformation law

$$\Gamma_a \to S\Gamma_a S^{-1} + Sd_a S^{-1}, \tag{4·31}$$

or equivalently,

$$a_{abc} \to a_{abc} - \frac{1}{g} \operatorname{tr} [\gamma_{bc} S d_a S^{-1}]. \tag{4·32}$$

The C_{abc} (4·20) are invariants under this group.

Finally, we will write down some relations that will be required in the following work. We will introduce a set of four real linearly independent reference vectors at each space-time point

$$h^\mu_a \quad (\mu = 0, 1, 2, 3 \text{ are labels for the four vectors}) \tag{4·33}$$

which are supposed to be smoothly varying functions of space-time. We define their scalar products

$$g^{\mu\nu} = h^\mu_a h^{a\nu}; \tag{4·34}$$

the matrix $g^{\mu\nu}$ has an inverse $g_{\mu\nu}$, because of the linear independence of the vectors. These two matrices can be used to raise and lower the Greek indices, while components with Greek indices can be assigned to any vector or tensor by projection onto the reference vectors:

$$v_a = h^\mu_a v_\mu. \tag{4·35}$$

From (4·34) we obtain

$$\eta_{ab} = h^\mu_a h_{b\mu}. \tag{4·36}$$

We extend the definition of the generalized derivative by requiring it to be zero for h^μ_a, that is, we define quantities $\Gamma_{\mu\nu}{}^\rho$ by

$$D_a h_b{}^\rho = d_a h_a{}^\rho - C_{ba}{}^\rho + \Gamma_{ba}{}^\rho = 0 \tag{4·37}$$

and then define
$$D_a v^\mu = d_a v^\rho + \Gamma_{\mu a}{}^\rho v^\mu. \tag{4·38}$$

All the expressions we have obtained in this section can now be rewritten with Greek indices (taking note of the fact that $d_\mu h_b{}^\nu$ is not zero). The commutator of the operators d_μ is

$$\left.\begin{aligned} [d_\mu, d_\nu] &= Y_{\mu\nu}{}^\rho d_\rho, \\ \text{where} \quad Y_{\mu\nu}{}^\rho &= X_{\mu\nu}{}^\rho + (d_\mu h^a_\nu - d_\nu h^a_\mu)\, h^\rho_a. \end{aligned}\right\} \tag{4·39}$$

Since the reference vectors (apart from linear independence and continuity requirements) are arbitrary, the formalism is invariant under a space-time variant group SL(4, R) of transformations

$$h^\mu_a \to G^\mu_\nu h^\nu_a. \tag{4·40}$$

It can easily be shown that $Y_{\mu\nu}{}^\rho$ transforms as a tensor under this group (i.e. it becomes $(G^{-1})^\alpha_\mu (G^{-1})^\beta_\nu G^\rho_\gamma Y_{\alpha\beta}{}^\gamma$) provided

$$d_\rho G^\mu_\nu - d_\nu G^\mu_\rho = 0. \tag{4·41}$$

If this restriction is imposed on the transformations, then we can choose the $X_{\mu\nu}{}^\rho$ so that $Y_{\mu\nu}{}^\rho = 0$, for it will then remain zero under the group (4·40). The d_μ then commute and we can identify them as differential operators in terms of a coordinate system: ∂_μ. The restriction (4·41) is then $\partial_\rho G^\mu_\nu - \partial_\nu G^\mu_\rho$ so that G^μ_ν has the form $\partial_\nu y^\mu$ for some quantities y^μ. The transformations (4·40) are then *general coordinate transformations* and y^μ are the new coordinates obtained by the transformation G^μ_ν. The transformation law of the $\Gamma_{\mu\nu}{}^\rho$ (4·37) is found to be

$$\Gamma_{\mu\nu}{}^\rho \to (G^{-1})^\rho_\nu [(G^{-1})^\alpha_\mu \Gamma_{\alpha\beta}{}^\gamma G^\rho_\gamma + (G^\rho_\alpha \partial_\beta G^{-1\alpha}_{\ \ \mu})] \tag{4·42}$$

which identifies them as components of an affine connexion, or equivalently, as auxiliary fields for the 'Yang–Mills' group (4·39). From $D_\mu g_{\rho\sigma} = 0$ (which follows from (4·34)) we find the *symmetric* part is given by

$$\tfrac{1}{2}(\Gamma_{\mu\nu}{}^\rho + \Gamma_{\nu\mu}{}^\rho) = \left\{ {}^{\rho}_{\mu\nu} \right\} = \tfrac{1}{2} g^{\rho\sigma}(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} + T_{\nu\sigma\mu} + T_{\mu\sigma\nu}) \tag{4·43}$$

where the skew part will be written

$$\tfrac{1}{2}(\Gamma_{\mu\nu}{}^\rho - \Gamma_{\nu\mu}{}^\rho) = \tfrac{1}{2} T_{\mu\nu}{}^\rho, \tag{4·44}$$

which is a tensor under (4·40).

The commutator $[D_\mu, D_\nu]$ is found to be, for a vector,

$$[D_\mu, D_\nu]\, v_\rho = R_{\mu\nu\rho}{}^\sigma v_\sigma + T_{\mu\nu}{}^\sigma D_\sigma v_\rho, \tag{4·45}$$

where
$$R_{\mu\nu\rho}{}^\sigma = \partial_\nu \Gamma_{\rho\mu}{}^\sigma - \partial_\mu \Gamma_{\rho\nu}{}^\sigma + \Gamma_{\rho\mu}{}^\lambda \Gamma_{\lambda\nu}{}^\sigma - \Gamma_{\rho\nu}{}^\lambda \Gamma_{\lambda\mu}{}^\sigma, \tag{4·46}$$

and for a spinor (rank 1 with lower index),

$$[D_\mu, D_\nu]\, \psi = \tfrac{1}{2} R_{\mu\nu\alpha\beta} \gamma^{\alpha\beta} \psi + T_{\mu\nu}{}^\sigma D_\sigma \psi \tag{4·47}$$

where
$$\tfrac{1}{2} R_{\mu\nu\alpha\beta} \gamma^{\alpha\beta} = \partial_\nu \Gamma_\mu - \partial_\mu \Gamma_\nu + [\Gamma_\mu, \Gamma_\nu]. \tag{4·48}$$

The Jacobi identify for the operators D_μ,

$$[D_\mu[D_\rho, D_\sigma]] + [D_\rho[D_\sigma, D_\mu]] + [D_\sigma[D_\mu, D_\rho]] = 0, \tag{4·49}$$

leads to the properties: $$D_{(\lambda} R_{\mu\nu)\rho}{}^{\sigma} = -T_{(\lambda\mu}{}^{\alpha} R_{\nu)\alpha\rho}{}^{\sigma}, \tag{4·50}$$

$$R_{(\lambda\mu\nu)}{}^{\sigma} = -D_{(\lambda} T_{\mu\nu)}{}^{\sigma} - T_{(\lambda\mu}{}^{\alpha} T_{\nu)\alpha}{}^{\sigma}. \tag{4·51}$$

With the identification of (4·40) as general coordinate transformations and of $\Gamma_{\mu\nu}{}^{\rho}$ as an affine connexion, the $g_{\mu\nu}$ become the components of the metric. Any field equations satisfied by the field $a_{\mu\nu\rho}$ in (4·15) will give rise to equations to be satisfied by the metric, so that the metric must now be interpreted as a physical field, as in general relativity. Thus the introduction of the coupling (4·13) together with the requirement that the d_a be linear combinations of the ∂_μ, necessitates the introduction of a curved space-time. In the next section we will investigate how the BW spin-2 theory is related to Einstein's theory.

5. *Non-linear spin-2 theory*. We now show in what sense the generalized Bargmann–Wigner spin-s theory

$$\gamma^{\mu E}{}_{A_1} D_\mu \psi_{EA_2\ldots A_{2s}} = m\psi_{A_1\ldots A_{2s}} \tag{5·1}$$

gives rise to the flat space theory of sections 1 and 2 under the limiting process $g \to 0$ where g is the coupling constant in (4·15). The limiting process involved is not quite a straightforward concept, since in the process we expect the space time to become flattened out, the coordinate system to be changed to a Euclidean coordinate system $(g_{\mu\nu} \to \eta_{\mu\nu})$, the reference vectors h^a_μ to become aligned with these Euclidean coordinates, and the matrices γ^μ to become constant matrices, as the coupling constant is changed. To facilitate the discussion of this limiting procedure we will take the γ_a in the curved space-time to be *constants*. This can be done by a transformation (4·30) on all spinor indices – in other words, we shall choose a 'special gauge' in order to freeze the group (4·30). With this choice of gauge equation (4·20) becomes simply

$$C_{bac} = g a_{abc} \tag{5·2}$$

so that now under the limit $g \to 0$ we have $D_a \to d_a (D_\mu \to \partial_\mu)$ for quantities with spinor indices or Latin tensor indices.

We will also accompany an infinitesimal change δg in the coupling constant by a change in the reference vectors so that a^c_a defined by

$$h_a{}^{\rho} = \delta_a{}^{\rho} + g a_a{}^{\rho} \tag{5·3}$$

is unchanged. That is, we accompany a change δg in the coupling constant by a change

$$\delta h_a{}^{\rho} = \tfrac{1}{2}\delta g a_a{}^{\rho} \tag{5·4}$$

in the reference vectors. Of course, (5·3) is not a covariant definition for either Lorentz rotations or general coordinate transformations, but $a_a{}^{\rho}$ has well-defined transformation properties. We shall define quantities $a_{\mu\sigma} = h^a_\mu a_a{}^{\rho} g_{\rho\sigma}$, etc., as if $a_{\mu\sigma}$ were a tensor, for convenience of notation. We then find that by the invariance of $\eta_{ab} = h^\mu_a h^\nu_b g_{\mu\nu}$ under (5·4), the metric components must undergo a change

$$\delta g_{\rho\sigma} = -\tfrac{1}{2}\delta g(a_{\rho\sigma} + a_{\sigma\rho}). \tag{5·5}$$

In general a change of the form (5·4) cannot be brought about by a coordinate transformation (this would require that $\partial_\mu a_\nu{}^\rho - \partial_\nu a_\mu{}^\rho$ be zero), so the change (5·5) is a change in the geometrical shape of the space-time, accompanied by a change of the coordinate system. This can be seen by the fact that if g is made zero by a series of changes (5·4) then $h_a{}^\rho$ becomes $\delta_a{}^\rho$ so that the metric components become the flat space metric $\eta_{\mu\nu}$.

We will define the quantities $A_{\mu\nu}$ by

$$g_{\mu\nu} = \eta_{\mu\nu} + gA_{\mu\nu}. \tag{5·6}$$

$A_{\mu\nu}$ is of course not a tensor for coordinate transformations. Under the limiting process described above it *remains finite*. In fact, by putting g infinitesimal in (5·3) we see that, in the limit, $A_{\mu\nu}$ coincides with $-\frac{1}{2}(a_{\mu\nu}+a_{\nu\mu})$. It will be shown that the limiting procedure described above uncouples the gravitational field from other fields and from itself, while leaving it finite, so that in flat space-time we are left with a linear spin-2 theory which is just the BW spin-2 theory.

That (5·1) becomes the flat space BW theory for spin s under the limiting procedure described above is now apparent. The metric becomes the flat space metric $\eta_{\mu\nu}$, the γ^μ become constant matrices (equal to the γ^a) and D_μ becomes ∂_μ for spinors.

With (5·2) and the definition (5·3) the quantities $\Gamma_{b\mu}{}^\rho$ of (4·37) are just

$$\Gamma_{b\mu}{}^\rho = g[a_{b\mu}{}^\rho - \tfrac{1}{2}\partial_\mu a_b{}^\rho] \tag{5·7}$$

so that they vanish in the limit and we can therefore write $D_\mu \to \partial_\mu$ for *all* quantities (spinors, tensors with Latin indices and tensors with Greek indices).

We now come to the problem of setting up the spin-2 theory of section 2 in the curved space. The spinor ψ_{ABCD} can still be expressed in terms of a real tensor $A_{\mu\nu\rho\sigma}$ with the symmetry properties (2·19–22), the only difference being that in the definition of the dual of a skew-symmetric pair of indices $\epsilon_{\mu\nu\rho\sigma}$ is replaced by $h\epsilon_{\mu\nu\rho\sigma}$ where h is the determinant of the 4×4 matrix h^a_μ ($h\epsilon_{\mu\nu\rho\sigma} = h^a_\mu h^b_\nu h^c_\rho h^d_\sigma \epsilon_{abcd}$). From (5·1) on ψ_{ABCD} (with $m=0$) we get

$$D_{(\mu}A_{\nu\rho)\sigma\lambda} = 0 \tag{5·8}$$

which does not enable us to define potentials by (2·26). The potentials must be defined in such a way that (2·26) is recovered in the limit $g\to 0$. This problem is already solved in that the formalism of section 4 yields all the equations for a flat space BW spin-2 theory, with primary and secondary potentials as described in section 2, *if we identify the auxiliary fields $a_{\mu\nu\rho}$ themselves as the potentials of the spin-2 field.*

According to (4·50) and (4·51), the Riemann tensor $R_{\mu\nu\rho\sigma}$ satisfies the cyclic symmetry $R_{(\mu\nu\rho)\sigma} = 0$ in common with $A_{\mu\nu\rho\sigma}$, and also satisfies the equation (5·8), *provided* the 'torsion tensor' $T_{\mu\nu}{}^\rho$ is zero (the assumption that $T_{\mu\nu}{}^\rho$ is zero also solves the problem of potentials in the generalized spin-1 theory, in that we then have

$$D_{(\mu}f_{\rho\sigma)} = \partial_{(\mu}f_{\rho\sigma)} \tag{5·9}$$

for a skew-symmetric rank 2 tensor, so that we can define A_μ by

$$f_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu \tag{5·10}$$

and for any vector $\partial_\mu A_\nu - \partial_\nu A_\mu = D_\mu A_\nu - D_\nu A_\mu$ if the torsion is zero). Defining $B_{\mu\nu\rho}$ by

$$\Gamma_{\mu\nu\rho} = gB_{\mu\nu\rho} \tag{5·11}$$

then $B_{\mu\nu\rho}$ remains finite in the limit $g \to 0$, and with vanishing torsion is symmetric in $(\mu\nu)$. From (4·43) we then obtain

$$B_{\mu\nu\rho} = \tfrac{1}{2}(\partial_\nu A_{\rho\mu} + \partial_\mu A_{\nu\rho} - \partial_\rho A_{\mu\nu}) \tag{5·12}$$

with $A_{\mu\nu}$ given by (5·6). This equation has precisely the same form as equation (2·35). Moreover, defining

$$A_{\mu\nu\rho\sigma} = \frac{2}{g} R_{\mu\nu\rho\sigma} \tag{5·13}$$

which also has a finite limit, we obtain from (4·46)

$$A_{\mu\nu\rho}{}^{\sigma} = 2(\partial_\nu B_{\rho\mu}{}^{\sigma} - \partial_\nu B_{\rho\nu}{}^{\sigma}) + 2g(B_{\rho\mu}{}^{\lambda} B_{\lambda\nu}{}^{\sigma} - B_{\rho\nu}{}^{\lambda} B_{\lambda\mu}{}^{\sigma}) \tag{5·14}$$

which in the limit gives just (2·36). Therefore, if we identify this tensor as the one occurring in the BW spin-2 theory (5·8), then $B_{\rho\mu\sigma}$ are the generalizations of the potentials of (2·35), and the secondary potentials are identified as the $A_{\mu\nu}$ of (5·6). The potentials $A_{\mu\rho\sigma}$ of section 2 are then just

$$A_{\mu\rho\sigma} = \partial_\sigma A_{\rho\mu} - \partial_\rho A_{\sigma\mu}. \tag{5·15}$$

Taking the limit of (5·7) then gives

$$A_{\nu\mu\rho} = 2a_{\nu\mu\rho} + \partial_\nu C_{\mu\rho}, \tag{5·16}$$

where $C_{\mu\rho} = \tfrac{1}{2}(a_{\rho\mu} - a_{\mu\rho})$, so that the basic Yang–Mills fields $2a_{\nu\mu\rho}$ of (4·15) differ from the potentials $A_{\nu\mu\rho}$ of the spin-2 field only by a transformation of the kind described in the paragraph following equation (2·28).

The remaining symmetry property imposed on $A_{\mu\nu\rho\sigma}$ by the properties of ψ_{ABCD} is the traceless property which in curved space-time gives Einstein's 'empty space' equations

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\nu\sigma} = 0. \tag{5·17}$$

It is quite remarkable that Einstein's equations in this treatment have arisen from the subsidiary condition

$$\gamma_{5A}^{E} \psi_{EBCD} = \psi_{ABCD} \tag{5·18}$$

on the zero mass spin-2 field. The analogous condition on the spin-$\tfrac{1}{2}$ zero mass field leads to the 'left-handedness' of neutrinos and the consequent parity violation of the leptonic weak interactions.

Finally, we have to deal with the spin-2 field in *interaction* with other fields:

$$D^{\mu} A_{\mu\nu\rho\sigma} = g j_{\nu\rho\sigma}, \tag{5·19}$$

where we have identified the coupling constant f of (3·1) with g, since this is the constant that describes the strength of the coupling of the spin-2 field to other fields. Of course, the $A_{\mu\nu\rho\sigma}$ can no longer be proportional to the Riemann tensor since the Bianchi identity is not satisfied. Instead we define

$$B_{\mu\nu\rho\sigma} = \frac{2}{g} R_{\mu\nu\rho\sigma} \tag{5·20}$$

and identify $B_{\mu\nu\rho\sigma}$ as the tensor of (3·10), with $j_{\nu\sigma}$ some symmetric tensor constructed from the other fields present

$$B_{\mu\nu\rho\sigma} = A_{\mu\nu\rho\sigma} + g(g_{\mu\rho} j_{\nu\sigma} - g_{\nu\rho} j_{\mu\sigma} + g_{\nu\sigma} j_{\nu\rho} - g_{\nu\sigma} j_{\nu\rho}). \tag{5·21}$$

Then
$$B_{\mu\nu} = g^{\rho\sigma}B_{\rho\mu\nu\sigma} = -g(2j_{\mu\nu}+g_{\mu\nu}j), \tag{5·22}$$
where $j = j^\mu_\mu = g^{\mu\nu}j_{\mu\nu}$, or equivalently
$$R_{\mu\nu} = -g^2(j_{\mu\nu}+\tfrac{1}{2}g_{\mu\nu}j), \tag{5·23}$$
$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R = -g^2(j_{\mu\nu}-g_{\mu\nu}j), \tag{5·24}$$
which because of the Bianchi identity implies that $j_{\mu\nu}$ must satisfy
$$D^\mu(j_{\mu\nu}-g_{\mu\nu}j) = 0. \tag{5·25}$$
If $j_{\mu\nu}-g_{\mu\nu}j$ is identified as a multiple of the stress-energy tensor for the matter field then (5·24) is just Einstein's equation for the gravitational field in interaction.

From the contracted Bianchi identity
$$D^\mu R_{\mu\nu\rho\sigma} = D_\sigma R_{\rho\nu}-D_\rho R_{\mu\sigma} \tag{5·26}$$
the divergence of (5·21) gives just
$$j_{\nu\rho\sigma} = D_\rho j_{\sigma\nu}-D_\sigma j_{\rho\nu} \tag{5·27}$$
which generalizes (3·7) and identifies the 'currents' $j_{\nu\rho\sigma}$. Using (5·22) to eliminate the $j_{\mu\nu}$ from (5·21) we obtain the generalization of (3·22):
$$A_{\mu\nu\rho\sigma} = \frac{2}{g}[R_{\mu\nu\rho\sigma}+\tfrac{1}{2}(g_{\mu\rho}R_{\nu\sigma}-g_{\mu\sigma}R_{\nu\rho}+g_{\nu\sigma}R_{\mu\rho}-g_{\nu\rho}R_{\mu\sigma})-\tfrac{1}{6}R(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})] \tag{5·28}$$
which identifies the tensor $C_{\mu\nu\rho\sigma}$ in
$$A_{\mu\nu\rho\sigma} = \frac{2}{g}C_{\mu\nu\rho\sigma} \tag{5·29}$$
as the Weyl tensor (20). The Weyl tensor, as well as possessing all the symmetries required for $A_{\mu\nu\rho\sigma}$, can be shown to satisfy $D_\mu D_\nu C_{\mu\nu\rho\sigma} = 0$, so that
$$D^\mu j_{\mu\rho\sigma} = 0 \tag{5·30}$$
which generalizes (3·5) and completes the discussion of the generalization of section 3 in the presence of the self-coupling of the spin-2 field.

The fields $h_a{}^\mu$ that we introduced are of course just the usual 'vierbein' fields (21) that are often made use of in general relativity. In fact our formalism is invariant under the usual 'Lorentz rotations of the vierbein' which are transformations
$$v^a \to \Lambda^a_b v^b$$
with *space-time dependent* Lorentz matrices, provided the a_{abc} are regarded as auxiliary fields for this group also. The C_{abc} are not tensors for this generalized Lorentz group. It is interesting to compare the present work with that of Sciama(22), who takes the vierbein rotations as the crucial Yang–Mills group and works in a curved space-time from the outset. A Lagrangian hR is varied with respect to the $h_a{}^\mu$ and $a_{\mu a}{}^b$ independently. For variation with respect to the $h_a{}^\mu$ we get Einstein's equations, while with respect to the $a_{\mu a}{}^b$ we get, in the presence of matter possessing spin, a relation which implies the non-vanishing of the torsion. This is to be contrasted with the present theory where we have abandoned the Lagrangian approach, relying instead on

Bargmann–Wigner equations, and identified the $a_{\mu a}{}^b$ as the gravitational field rather than an independent entity. Also, the vanishing of the torsion tensor is an essential aspect of the formulation. Any Lagrangian approach to the ideas presented in the present work would be expected to be quite complex, on account of our reliance on Bargmann–Wigner theory.

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