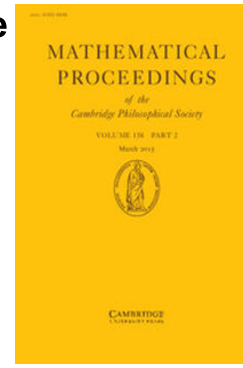

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E. A. Lord

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The Dirac spinor in six dimensions

BY E. A. LORD
Department of Mathematics, King's College, University of London

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Abstract. The spinor representations of the rotation group in a six-dimensional space with indefinite metric are shown to be four-component spinors, which become the usual Dirac spinors when the formalism is restricted to a four-dimensional subspace. Eriksson's work on the five-dimensional Lorentz group is found to result from a restriction of the six-dimensional treatment to a five-dimensional subspace, and the algebraic significance of Eriksson's work is thereby clarified.

1. *Introduction.* Some aspects of the relationship between the Clifford algebras and the spinor representations of the complex orthogonal groups are well known. The classic work on this aspect of spinors is Brauer and Weyl's 'Spinors in n Dimensions'(2). The physical importance of this work is apparent in Dirac's theory of the electron (3), the 4-component wave-function for a spin $\frac{1}{2}$ particle being a spinor whose transformation law under Lorentz rotations and reflexions is derivable from the structure of the Clifford algebra C_4.

An aspect of spinor representations is developed in the present paper which has not previously been dealt with in general terms in the literature. The Weyl equation (see, for instance, Roman's *Theory of Elementary Particles* (10)) deals with a 2-component spinor representation of the proper Lorentz group, which has no analogue in the work of Brauer and Weyl. The connexion between this 2-component representation and the theory of the quaternion algebra C_2 is given by Rastall (9), Ellis (5), and Lord (7). The two component spinor representation of the proper Lorentz group, and the derivation of its transformation law from the structure of C_2, is in fact a special case of a more general procedure whereby a 2^ν-component spinor representation of the rotations in a certain real $(2\nu+2)$-dimensional space with indefinite metric can be obtained from the structure of the Clifford algebra $C_{2\nu}$. The aim of the present paper is to investigate these representations, with particular reference to the 4-component spinor representations of the group of proper rotations in a six-dimensional space with metric $(+++++-)$.

2. *The 'basic' spinor representations.* The Clifford algebra C_n is defined to be the algebra generated by a set of n elements e_μ ($\mu = 1, \dots n$) which anticommute with each other and have unit square

$$\tfrac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) = \delta_{\mu\nu} \quad (\mu, \nu = 1, \dots n). \tag{2·1}$$

When n is even ($n = 2\nu$, say), there is only one irreducible representation, which is 2^ν-dimensional, and when n is odd ($n = 2\nu+1$), there are two inequivalent irreducible representations, both of dimension 2^ν (Boerner, *The Representations of Groups* (1)). A linearly independent basis for the algebra is the set of 2^n elements

$$1, \quad e_\mu, \quad \underset{(\mu<\nu)}{e_\mu e_\nu}, \quad \underset{(\mu<\nu<\rho)}{e_\mu e_\nu e_\rho}, \quad \ldots, (e_1 e_2 \ldots e_n),$$

which, for n even, are all traceless in any representation, with the exception of the unit element.

The skew-symmetrized products

$$e_{\mu\nu} = \tfrac{1}{2}(e_\mu e_\nu - e_\nu e_\mu)$$

are found to satisfy

$$[e_{\mu\nu}, e_{\rho\sigma}] = 2(\delta_{\mu\sigma} e_{\nu\rho} - \delta_{\mu\rho} e_{\nu\sigma} + \delta_{\nu\rho} e_{\mu\sigma} - \delta_{\nu\sigma} e_{\mu\rho}), \tag{2·2}$$

so that the quantities

$$G_{\mu\nu} = \tfrac{1}{2} e_{\mu\nu} \tag{2·3}$$

formed from an *irreducible* representation of the e_μ, are the infinitesimal generators of a 2^ν-dimensional representation of $SO(n, c)$. The $G_{\mu\nu}$ are linearly independent, so that the subspace of C_n consisting of the elements $G_{\mu\nu}$, considered as a Lie ring, is isomorphic to the Lie ring of $SO(n, c)$ (Boerner (1)). We shall call such a representation of $SO(n, c)$ a *basic* spinor representation. We shall show that it is a faithful representation, and consequently the basic spinor representation is a representation of the whole group, not just a subgroup. There are two basic spinor representations when n is odd, but these are in fact equivalent; when n is even there is one basic spinor representation, and, as we shall see, it is *reducible*.

The basic spinor representations, introduced here from an infinitesimal viewpoint, are the ones discussed by Brauer and Weyl. For any orthogonal matrix

$$\lambda_{\mu\nu} \, (\mu, \nu = 1, \ldots n),$$

and any irreducible representation e_μ of the generators of C_n, there exists a matrix S such that

$$e_\mu = \lambda_{\mu\nu} S e_\nu S^{-1}. \tag{2·4}$$

The matrix S is determined only up to a numerical factor, so we may impose the restriction that S be unimodular. The matrix S is then determined to within a numerical factor ± 1. If n is odd a matrix S satisfying (2·4) exists only if $\lambda_{\mu\nu}$ is unimodular as well as orthogonal. The matrices S therefore give a representation of $SO(n, c)$ if n is odd, and of the full group $O(n, c)$ if n is even. Any matrix S determines the rotation $\lambda_{\mu\nu}$ uniquely, since

$$\lambda_{\mu\nu} = \left(\frac{1}{2^\nu}\right) \operatorname{tr}(S^{-1} e_\mu S e_\nu).$$

Hence we have established a faithful 2–1 representation of the group $SO(n, c)$ (and of $O(n, c)$ if n is even), which is in fact the basic spinor representation as defined above. To show this, we take an infinitesimal rotation

$$\lambda_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu} \quad (\epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \text{ infinitesimal}),$$
$$S = 1 + \tfrac{1}{2}\epsilon_{\mu\nu} G_{\mu\nu},$$

and the expression (2·4) then becomes

$$0 = \epsilon_{\mu\nu} e_\nu + \tfrac{1}{2}\epsilon_{\rho\sigma}[G_{\rho\sigma}, e_\mu].$$

Multiplying by e_μ and making use of the identity

$$e_\mu e_{\rho\sigma} e_\mu = (n-4)\, e_{\rho\sigma},$$

we obtain the solution (2·3) for the infinitesimal generators.

When n is odd, the two irreducible representations of C_n can be obtained from the irreducible representation of the even Clifford algebra C_{n-1}. Thus, for instance, if $\gamma_\mu\,(\mu = 1 \dots 4)$ are the generators of the Dirac algebra C_4, the matrix

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$$

has unit square and anticommutes with the four generators. The matrices $\gamma_\mu\,(\mu = 1\dots 5)$ therefore satisfy

$$\tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \delta_{\mu\nu} \quad (\mu, \nu = 1, \dots 5), \tag{2·5}$$

and so we have formed an irreducible representation of C_5. The other irreducible representation is obtained by the substitution

$$\gamma_\mu \to -\gamma_\mu \quad (\mu = 1\dots 5). \tag{2·6}$$

The quantities
$$G_{\mu\nu} = \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \quad (\mu, \nu = 1, \dots 5) \tag{2·7}$$

are the infinitesimal generators of one of the 4-component basic spinor representations of $SO(5, c)$. As we see, the generators are unaffected by the change of representation (2·6) of C_5, and so the two basic spinor representations of $SO(6, c)$ obtained from the two inequivalent irreducible representations of C_5 are in fact equivalent. It is easy to see that this is true for any odd n—there is no loss of generality involved in our choice $n = 5$. The basic spinor representation $SO(5, c)$, as defined here, has been thoroughly discussed by Pauli (8).

The $G_{\mu\nu}$ given by (2·8) are linearly independent, so that the ten-dimensional subspace of C_5 given by the quantities $G_{\mu\nu}$, considered as a Lie ring, is isomorphic to the Lie ring of $SO(5, c)$. In general, the Lie ring defined from the $\tfrac{1}{2}n(n-1)$ elements $\tfrac{1}{2}e_{\mu\nu}\,(\mu, \nu = 1 \dots n)$ of C_n is isomorphic to the Lie ring of $SO(n, c)$. Thus the group of transformation matrices defined from the infinitesimal generators $\tfrac{1}{2}e_{\mu\nu}$ is a faithful representation of the group $SO(n, c)$. The basic spinor representations therefore give a representation of the whole group, not just a subgroup.

The representations of C_5 generated by γ_μ and by $-\gamma_\mu (\mu = 1, \dots 5)$ are inequivalent, so that there is no matrix S with the property

$$\gamma_\mu = -S\gamma_\mu S^{-1} \quad (\mu = 1, \dots 5).$$

Reflexions in the five-dimensional space are therefore not represented by the basic spinors, and this is true for any odd-dimensional space. In an even-dimensional space, however, reflexions are included, so that the basic spinors give a representation of the full orthogonal group $O(n, c)$. For instance, in four dimensions the equations

$$\gamma_i = S\gamma_i S^{-1} \quad (i = 1, 2, 3),$$
$$\gamma_4 = -S\gamma_4 S^{-1}$$

representing a reflexion parallel to the fourth axis, have the solution

$$S = \gamma_4\gamma_5.$$

3. *Decomposition of the basic spinor representations.* We have already stated that the basic spinor representations of $SO(n,c)$, for n even, are reducible. We shall consider here the reduction of the basic 8-component spinor representation of $SO(6,c)$ into constituent 4-component representations. The procedure is easily generalized to apply to any even-dimensional space. An irreducible representation of the Clifford algebra C_6 can be constructed from a given irreducible representation of C_5. The $\gamma_\mu (\mu = 1, \ldots 5)$ of an irreducible representation of C_5 satisfy (2·5), so that the quantities e_α $(\alpha = 1, \ldots 6)$ defined by

$$\left.\begin{aligned} e_\mu &= (-i)\begin{pmatrix} & \gamma_\mu B \\ -B^{-1}\gamma_\mu & \end{pmatrix} \quad (\mu = 1, \ldots 5), \\ e_6 &= \begin{pmatrix} & B \\ B^{-1} & \end{pmatrix}, \end{aligned}\right\} \tag{3·1}$$

satisfy $$\tfrac{1}{2}(e_\alpha e_\beta + e_\beta e_\alpha) = \delta_{\alpha\beta} \quad (\alpha, \beta = 1, \ldots 6).$$

For the moment the matrix B is completely arbitrary. The e_α therefore give an irreducible representation of C_6. They can be written more concisely in terms of the sets of matrices a_α and $\bar{a}_\alpha$, defined as follows:

$$\left.\begin{aligned} a_\mu &= \gamma_\mu B, \quad \bar{a}_\mu = -B^{-1}\gamma_\mu \quad (\mu = 1, \ldots 5), \\ a_6 &= iB, \qquad \bar{a}_6 = iB^{-1}. \end{aligned}\right\} \tag{3·2}$$

The generator e_α of C_6 can then be written

$$e_\alpha = (-i)\begin{pmatrix} & a_\alpha \\ \bar{a}_\alpha & \end{pmatrix} \quad (\alpha = 1, \ldots 6). \tag{3·3}$$

With $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ we then find

$$e_7 = ie_1e_2e_3e_4e_5e_6 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

and $$e_{\alpha\beta} = \tfrac{1}{2}(e_\alpha e_\beta - e_\beta e_\alpha) = -\frac{1}{2}\begin{pmatrix} a_\alpha\bar{a}_\beta - a_\beta\bar{a}_\alpha & \\ & \bar{a}_\alpha a_\beta - \bar{a}_\beta a_\alpha \end{pmatrix}.$$

This latter expression shows that the infinitesimal generators of the 8-component spinor representation of $SO(6,c)$ is the direct sum of two 4-component representations. The basic spinor can be written

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

where ϕ and χ are four component spinors which transform according to representations with infinitesimal generators

$$\left.\begin{aligned} G^{(1)}_{\alpha\beta} &= -\tfrac{1}{4}(a_\alpha\bar{a} - a_\beta\bar{a}_\alpha) \\ \text{and} \quad G^{(2)}_{\alpha\beta} &= -\tfrac{1}{4}(\bar{a}_\alpha a_\beta - \bar{a}_\beta a_\alpha), \end{aligned}\right\} \tag{3·4}$$

respectively. In terms of the chosen representation γ_μ of C_5 these generators take the form

$$\left.\begin{aligned} G^{(1)}_{\mu\nu} &= \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \quad (\mu = 1, \dots 5), \\ G^{(1)}_{\mu 6} &= -\tfrac{1}{2}i\gamma_\mu, \\ G^{(2)}_{\mu\nu} &= \tfrac{1}{4}B^{-1}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)B, \\ G^{(2)}_{\mu 6} &= \tfrac{1}{2}iB^{-1}\gamma_\mu B. \end{aligned}\right\} \tag{3·5}$$

In either case, the set of 15 generators is linearly independent, so that the Lie ring defined from the $G_{\alpha\beta}$ is isomorphic to the Lie ring of $SO(6, c)$. More generally, the subspace of $\frac{1}{2}n(n+1)$ base elements $G_{\mu\nu} = \frac{1}{4}(e_\mu e_\nu - e_\nu e_\mu)$, $G_{\mu n+1} = -\frac{1}{2}ie_\mu$ $(\mu = 1, \dots n)$ of an irreducible representation of C_n, considered as a Lie ring, is isomorphic to the Lie ring of the rotation group $SO(n+1, c)$ (Boerner (1)). The case $n = 3$ is an exception, which we shall discuss in section 4.

The 15 generators in either case are just the traceless base elements of an irreducible representation of the Dirac algebra C_4, and consequently any (4×4) matrix can be written as a linear combination of them. The Lie ring of the group $SO(6, c)$ in this representation therefore consists of all traceless (4×4) matrices. Thus the lie ring of $SO(6, c)$ is isomorphic to that of the unimodular group $SL(4, c)$.

We may also consider the behaviour of the basic 8-component spinor under reflexions in the six-dimensional space. The decomposition into separate 4-component quantities then no longer occurs, of course. For instance, the reflexion parallel to the sixth axis is given by the matrix

$$S = e_6 e_7 = \begin{pmatrix} & -B \\ B^{-1} & \end{pmatrix}$$

and so, under this reflexion the 8-component spinor ψ indergoes the transformation

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \rightarrow \begin{pmatrix} -B\chi \\ B^{-1}\phi \end{pmatrix}.$$

If we set $B = 1$, the analogy with the behaviour of the 4-component Dirac spinor (in the Weyl representation) under reflexions in four dimensions becomes at once apparent.

4. *Quaternions.* The linear independence of the generators $G^{(1)}_{\alpha\beta}$ and $G^{(2)}_{\alpha\beta}$ shows that the Lie ring generated by these quantities is in fact isomorphic to the Lie ring of $SO(6,c)$ This is not true for the corresponding case for $SO(4, c)$. We may define an irreducible representation of C_4 in terms of the quaternions, in just the same way that the generators e_α of C_6 were constructed from the generators of the Dirac algebra. Defining

$$S_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad S_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \tag{4·1}$$

and taking an arbitrary (2×2) matrix B, we have a representation γ_μ $(\mu = 1\dots 4)$ of the generators of C_4:

$$\gamma_\mu = (-i)\begin{pmatrix} & \sigma_\mu \\ \overline{\sigma}_\mu & \end{pmatrix},$$

where
$$\left.\begin{array}{ll}\sigma_i = S_i B, & \bar{\sigma}_i = -B^{-1}S_i \quad (i = 1,2,3),\\ \sigma_4 = iB, & \bar{\sigma}_4 = iB^{-1}.\end{array}\right\} \tag{4·2}$$
We then find
$$\gamma_{\mu\nu} = \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) = -\frac{1}{2}\begin{pmatrix}\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu & \\ & \bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu\end{pmatrix},$$
so that, in this representation, the 4-component basic spinor representation ψ of $SO(4,c)$ can be written $\begin{pmatrix}\phi\\ \chi\end{pmatrix}$, where the constituent 2-component spinors ϕ and χ of $SO(4,c)$ transform according to representations with generators
$$G^{(1)}_{ij} = \tfrac{1}{4}(S_iS_j - S_jS_i) = \tfrac{1}{2}iS_k \quad (i,j,k \text{ an even permutation of } 1,\ 2,\ 3),$$
$$G^{(1)}_{k4} = -\tfrac{1}{2}iS_k \quad (k = 1,2,3),$$
and
$$G^{(2)}_{ij} = \tfrac{1}{2}iB^{-1}S_kB,$$
$$G^{(2)}_{k4} = \tfrac{1}{2}iB^{-1}S_kB,$$
respectively. These sets are *not* linearly independent. Consider an infinitesimal element $\lambda_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu} (\mu,\nu = 1,\ldots 4)$ of $SO(4,c)$. The spinors ϕ and χ transform to $p\phi$ and $q\chi$ where
$$p = 1 + (\tfrac{1}{2}\epsilon_{ijk}\epsilon_{ij} - \epsilon_{k4})\,\tfrac{1}{2}iS_k,$$
$$q = 1 + (\tfrac{1}{2}\epsilon_{ijk}\epsilon_{ij} + \epsilon_{k4})\,\tfrac{1}{2}iB^{-1}S_kB.$$
Thus neither of the matrices p and q determines uniquely the $\epsilon_{\mu\nu}$. However, if ϵ_{ij} is *real*, ϵ_{k4} *imaginary*, then the infinitesimal rotation $\epsilon_{\mu\nu}$ is determined if either p or q is given.

The subgroup of $SO(4,c)$ for which λ_{ij}, λ_{44} are real, λ_{i4} imaginary $(i,j = 1,2,3)$, is represented *faithfully* by either p or q. This subgroup is the proper Lorentz group. In this case, choosing $B = 1$, we have
$$p = (q^\dagger)^{-1},$$
so that the spinor χ transforms according to the matrix *adjoint* to the transformation matrix for σ. We have then arrived at the usual decomposition of the 4-component Dirac spinor representation of the proper Lorentz group into its two constituent 2-component spinors.

5. *Spin-tensors in six dimensions.* We now return to the consideration of the 4-component spinors in six dimensions, introduced in section 3. The (8×8) matrix S representing a rotation of $SO(6,c)$ has the form $\begin{pmatrix}p & \\ & q\end{pmatrix}$, the infinitesimal generators for the matrices p and q being given by (3·5). The basic 8-component spinor has the form $\begin{pmatrix}\phi\\ \chi\end{pmatrix}$. Thus the two types of 4-component spinors have the transformation laws
$$\phi \to p\phi,$$
$$\chi \to q\chi.$$
We may call these spinors 'covariant of the first type' and 'contravariant of the second type', and define quantities α and β with transformation laws
$$\alpha \to \alpha p^{-1},$$
$$\beta \to \beta q^{-1},$$

which we call 'contravariant of the first type' and 'covariant of the second type', respectively. A spin-tensor in the six-dimensional complex space can then be defined as any quantity with mixed tensor and spinor transformation laws under operations of $SO(6, c)$.

The transformation law of the e_α given by (3·3),

$$e_\alpha = \lambda_{\alpha\beta} S e_\beta S^{-1},$$

decomposes as follows

$$\left.\begin{aligned} a_\alpha &= \lambda_{\alpha\beta}\, p a_\beta q^{-1}, \\ \bar{a}_\alpha &= \lambda_{\alpha\beta}\, q \bar{a}_\beta p^{-1}. \end{aligned}\right\} \tag{5·1}$$

These equations may now be considered as the definition of the transformation laws of two spin-tensors a_α and $\bar{a}_\alpha$, and also as the statement that their components remain invariant. These components are fixed by (3·2) from a given fixed (invariant) representation of C_5. From (5·1) we obtain

$$\lambda_{\alpha\beta} = -\tfrac{1}{4}\operatorname{tr}(p^{-1} a_\alpha q \bar{a}_\beta) = -\tfrac{1}{4}\operatorname{tr}(q^{-1}\bar{a}_\alpha\, p a_\beta), \tag{5·2}$$

so that the rotation $\lambda_{\alpha\beta}$ is not in general uniquely determined by either p or q alone, but only by specifying both p and q. Thus the transformation matrices p and q of the 4-component spinors ϕ and χ do not in general give a faithful representation of the whole of $SO(6, c)$, only of a subgroup. The form of this subgroup depends on the choice of the matrix B.

The components of a_α and $\bar{a}_\alpha$ given by (3·2) satisfy

$$B\bar{a}_\alpha B = -a_\alpha + 2iC_\alpha B, \tag{5·3}$$

where C_α is a unit vector in the direction of the sixth axis. We wish to re-interpret this expression as a fully covariant equation for the operations of $SO(6, c)$. For a fixed vector C_α we must therefore re-interpret B as a rank two mixed spinor whose components have the transformation law

$$B \to pBq^{-1}. \tag{5·4}$$

The components of this spinor coincide with those of the arbitrarily chosen fixed matrix of (3·2) only in those reference frames in which the sixth axis is oriented along the vector C_α. These frames in the six-dimensional space we will call the 'special' frames. The special frames are transformed into each other by the subgroup of $SO(6, c)$ consisting of matrices of the form

$$\lambda_{\alpha\beta} = \begin{pmatrix} \lambda_{\mu\nu} & \\ & 1 \end{pmatrix} \quad \begin{matrix} (\alpha, \beta = 1, \ldots 6), \\ (\mu, \nu = 1, \ldots 5), \end{matrix}$$

i.e.: the subgroup $SO(5, c)$. For an infinitesimal rotation about C_α, the form (3·5) of the generators for p and q shows that

$$p = BqB^{-1},$$
$$B = pBq^{-1},$$

so that the spinor B is in fact invariant under these rotations.

We now define the spin-tensors γ_α and $\overline{\gamma}_\alpha$ as follows:

$$\left.\begin{aligned}\gamma_\alpha &= a_\alpha B^{-1},\\ \overline{\gamma}_\alpha &= B\overline{a}_\alpha \quad (\alpha = 1, \dots 6).\end{aligned}\right\} \tag{5·5}$$

These spin-tensors have the transformation laws

$$\left.\begin{aligned}\gamma_\alpha &\to \lambda_{\alpha\beta}\, p\gamma_\beta p^{-1},\\ \overline{\gamma}_\alpha &\to \lambda_{\alpha\beta}\, p\overline{\gamma}_\alpha p^{-1},\end{aligned}\right\} \tag{5·6}$$

and satisfy

$$\overline{\gamma}_\alpha = -\gamma_\alpha + 2iC_\alpha.$$

Unlike the $a_\alpha, \overline{a}_\alpha$, the quantities γ_α, $\overline{\gamma}_\alpha$ are not invariant, except under the operations of $SO(5, c)$ as defined above. In the special frames the components of γ_α coincide with those of the generators of C_5 from which the a_α are defined, and $\gamma_6 = i$. In the special frames,

$$\left.\begin{aligned}\overline{\gamma}_\mu &= -\gamma_\mu \quad (\mu = 1, \dots 5),\\ \overline{\gamma}_6 &= \gamma_6.\end{aligned}\right\}$$

The components of the spinor B are determined by those of the a_α, and the vector C_α. In the special frames we have

$$i = \gamma_6 = a_6 B^{-1} = C_\alpha a_\alpha B^{-1},$$

so therefore

$$B = -iC_\alpha a_\alpha. \tag{5·7}$$

This is a fully covariant equation, valid in all frames. The relation

$$-\tfrac{1}{2}(a_\alpha \overline{a}_\beta + a_\beta \overline{a}_\alpha) = \delta_{\alpha\beta}$$

implies

$$BC_\beta \overline{a}_\beta = -iC_\beta C_\beta = i,$$

so that

$$B^{-1} = -iC_\beta \overline{a}_\beta. \tag{5·8}$$

6. *Special matrices of the Dirac algebra.* Given any irreducible representation $\gamma_\mu (\mu = 1, \dots 5)$ of the Dirac algebra C_4, there exist matrices H, C and ϵ with the properties

$$\begin{aligned}\gamma_\mu^\dagger &= H^{-1}\gamma_\mu H,\\ \gamma_\mu^\times &= C^{-1}\gamma_\mu C,\\ \tilde{\gamma}_\mu &= \epsilon^{-1}\gamma_\mu \epsilon \quad (\mu = 1, \dots 5),\end{aligned}$$

where $\gamma_\mu^\dagger$, $\gamma_\mu^\times$ and $\tilde{\gamma}_\mu$ are respectively the Hermitian conjugate, complex conjugate and transpose of γ_μ. Defining

$$\begin{aligned}\overline{\gamma}_\mu &= -\gamma_\mu,\\ \overline{\gamma}_6 &= \gamma_6 = i,\end{aligned}$$

we can write

$$\left.\begin{aligned}\gamma_\alpha^\dagger &= -H^{-1}\overline{\gamma}_\alpha H,\\ \gamma_\alpha^\times &= -C^{-1}\overline{\gamma}_\alpha C,\\ \tilde{\gamma}_\alpha &= \epsilon^{-1}\gamma_\alpha \epsilon \quad (\alpha = 1, \dots 6).\end{aligned}\right\} \tag{6·1}$$

The matrix H may be chosen Hermitian, ϵ is skew-symmetric, and we may write $C = \epsilon \tilde{B}^{-1}$. A discussion of such matrices may be found in Roman's *Theory of Elementary Particles* (10). Now, if the equations (6·1) are to be re-interpreted as fully covariant relations between spin tensors of $SO(6,c)$ we must interpret H, C and ϵ as rank two spinors with the transformation laws

$$H \to pHp^{\dagger},$$
$$C \to pC(p^{\times})^{-1},$$
and
$$\epsilon \to p\epsilon\tilde{p}.$$

(i) Suppose the components of a_α and $\bar{a}_\alpha$ are determined by choosing the spinor B to coincide with ϵ in the special frames. Substituting ϵ for B in the expressions (3·5) for the infinitesimal generators, we find.

$$G^{(2)}_{\alpha\beta} = -\tilde{G}^{(1)}_{\alpha\beta}.$$

Thus, with this choice for B, the transformation matrices p and q for any rotation in the six-dimensional complex space satisfy

$$q^{-1} = \tilde{p}.$$

Thus the spinor B has the transformation law

$$B \to pB\tilde{p},$$

and therefore coincides with the spinor ϵ in any reference frame. With this choice for the spinor B, (5·2) becomes

$$\lambda_{\alpha\beta} = -\tfrac{1}{4}\operatorname{tr}(\tilde{p}\bar{a}_\alpha p a_\beta) = -\tfrac{1}{4}\operatorname{tr}(\tilde{q}a_\alpha q\bar{a}_\beta),$$

so that either p or q determine the rotation completely. The two-valued spinor representations of $SO(6,c)$ given by ϕ or χ are therefore in this case faithful representations of the whole group.

(ii) Suppose now we choose the spinor B so that it coincides with the spinor C in the special frames. Substituting C for B in (3·5) we obtain

$$G^{(2)}_{\mu\nu} = G^{(1)\times}_{\mu\nu} \quad (\mu,\nu = 1,\dots 5),$$
$$G^{(2)}_{\mu 6} = -G^{(1)\times}_{\mu 6}.$$

Thus, if we restrict the group $SO(6,c)$ to the subgroup $L(6)$ for which $\lambda_{\mu\nu}$, λ_{66} are *real*, $\lambda_{\mu 6}$ *imaginary*, we have

$$p^{\times} = q$$

for the representation matrices associated with a rotation in $L(6)$. With this restriction, the transformation law of the spinor B is

$$B \to pB(p^{\times})^{-1}$$

and consequently C and B are identical. The matrix $\lambda_{\alpha\beta}$ of a rotation in $L(6)$ is determined uniquely by the matrix p

$$\lambda_{\alpha\beta} = -\tfrac{1}{4}\operatorname{tr}(p^{-1}a_\alpha p^{\times}\bar{a}_\beta),$$

so that the four-component spinor representations in this case give a faithful representation of $L(6)$.

(iii) Similarly, we may identify the components of the spinors H and B in the special frames. In this case,

$$G^{(2)}_{\mu\nu} = -G^{(1)\dagger}_{\mu\nu} \quad (\mu,\nu = 1,\dots 5),$$
$$G^{(2)}_{\mu 6} = G^{(1)\dagger}_{\mu 6},$$

so that, *again for the subgroup* $L(6)$, we have

$$q^{-1} = p^{\dagger},$$

and the transformation laws of H and B are identical. The rotation $\lambda_{\alpha\beta}$ of $L(6)$ is given by

$$\lambda_{\alpha\beta} = -\tfrac{1}{4}\operatorname{tr}(p^{\dagger}\bar{a}_{\alpha}pa_{\beta}),$$

so that the 4-component spinor representations are *faithful* representations of the subgroup $L(6)$.

This third choice for the spinor B is the one we shall use in the remainder of the paper. We must then restrict the operations of $SO(6,c)$ to those of the subgroup $L(6)$. This subgroup corresponds to the group of rotations in a *real* six-dimensional space with signature $(+ + + + + -)$. The expression

$$\gamma_{\alpha}^{\dagger} = -B^{-1}\bar{\gamma}_{\alpha}B$$

is then a fully covariant relation between spin-tensors in six dimensions.,

7. *Relations between tensors and spinors.* If we restrict our attention to the subgroup $L(6)$ of $SO(6,c)$, then we may take the spinor B to satisfy

$$\gamma_{\alpha}^{\dagger} = -B^{-1}\bar{\gamma}_{\alpha}B. \tag{7·1}$$

In this case, for any rotation, the transformation laws of the elementary 4-component spinors are

$$\phi \to p\phi,$$
$$\chi \to (p^{\dagger})^{-1}\chi,$$

and we may also introduce quantities with the transformation laws

$$\alpha \to \alpha p^{-1},$$
$$\beta \to \beta p^{\dagger}.$$

The spin-tensors $G_{\alpha\beta} = -\frac{1}{4}(a_{\alpha}\bar{a}_{\beta} - a_{\beta}\bar{a}_{\alpha})$ are invariant under rotations; they have the transformation law

$$G_{\alpha\beta} \to \lambda_{\alpha\gamma}\lambda_{\beta\delta}pG_{\gamma\delta}p^{-1} = G_{\alpha\beta}.$$

The set of 16 matrices 1, $G_{\alpha\beta}$ span all (4×4) matrices, and therefore any rank two spinor Φ in the six-space with the transformation law

$$\Phi \to p\Phi p^{-1} \tag{7·2}$$

can be written as a linear combination of 1 and $G_{\alpha\beta}$

$$\Phi = I + \tfrac{1}{2}G_{\alpha\beta}\phi_{\alpha\beta},$$

where I is an invariant and $\phi_{\alpha\beta}$ is a rank two skew-symmetric tensor under operations of $L(6)$. On restricting the rotations further to real rotations in the five-dimensional subspace perpendicular to the sixth axis, we obtain

$$\Phi = I + (-\tfrac{1}{2}i\phi_{\mu 6})\gamma_{\mu} + (\tfrac{1}{4}\phi_{\mu\nu})\gamma_{\mu\nu} \quad (\mu, \nu = 1, \dots 5),$$

where $\gamma_{\mu\nu} = \frac{1}{2}(\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu})$. Restricting the rotations still further to a four-dimensional subspace, we obtain the well-known correspondence between a rank two Dirac spinor and a scalar, a vector, a skew-symmetric rank two tensor, a pseudovector and a pseudoscalar (Eddington(4)).

Suppose now the matrix of components of Φ is *traceless*—a condition which is invariant under the operation (7·2), we then have

$$\Phi = \tfrac{1}{2}G_{\alpha\beta}\phi_{\alpha\beta}. \tag{7·3}$$

We have therefore established a one-one correspondence between traceless rank two-spinors with the transformation law (7·2), and skew-symmetric tensors in the six-dimensional space.

Another correspondence between spinors and vectors is obtained as follows. For any representation $\gamma_\mu (\mu = 1 \dots 5)$ of the Dirac algebra, the matrices

$$\epsilon,\ \gamma_\mu\epsilon \quad \text{are skew-symmetric,}$$

and

$$\tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\,\epsilon \quad \text{are symmetric.}$$

Thus any skew-symmetric (4×4) matric c can be written

$$c = A_\alpha\gamma_\alpha\epsilon \quad (\alpha = 1, \dots 6). \tag{7·4}$$

This is a fully covariant equation under the operations of $L(6)$ provided we interpret A_α as a six-vector and c as a rank two spinor with the transformation law

$$c \to pc\tilde{p},$$

under which its skew-symmetry is preserved. Thus we have established a one-one correspondence between rank two skew-symmetric spinors and six vectors.

8. *Introduction of spinor indices*. So far we have suppressed all spinor indices. We find that by employing four types of spinor index (upper and lower, dotted and undotted), we can indicate the transformation law of any spin-tensor by the nature of its indices. We shall use capital latin letters to indicate spinor indices, the components of the transformation matrix p being written as $p_A{}^B$, which the components of the adjoint matrix $(p^\dagger)^{-1} = q$ will be written $p^{\dot{A}}{}_{\dot{B}}$. We then have four types of rank one spinor:

(i) covariant, ϕ_A, with transformation law $\phi \to p\phi$;
(ii) contravariant α^A, with transformation law $\alpha \to \alpha p^{-1}$;
(iii) covariant, dotted $\beta_{\dot{A}}$ with transformation law $\beta \to \beta p^\dagger$;
(iv) contravariant, dotted $\chi^{\dot{A}}$, with transformation law $\chi \to (p^\dagger)^{-1}\chi$.

A quantity with a covariant transformation law can be formed from another by 'contraction' on two spinor indices only if one is a superscript, one a subscript, and if they are either both dotted or both undotted. The quantities

$$\gamma_\alpha,\quad \bar{\gamma}_\alpha,\quad a_\alpha,\quad \bar{a}_\alpha,\quad B,\quad C \quad \text{and} \quad \epsilon,$$

then have the transformation laws implied by the following assignment of indices:

$$\gamma_{\alpha A}{}^B,\quad \bar{\gamma}_{\alpha A}{}^B,\quad a_{\alpha A\dot{B}},\quad \bar{a}_\alpha^{\dot{B}A},\quad B_{A\dot{B}},\quad C_A{}^{\dot{B}},\quad \epsilon_{AB}.$$

Given any mixed rank two spinor $a_{\dot{A}B}$, we can form an undotted rank two spinor b_{AB} from it by multiplication on the left by the spinor C

$$b_{AB} = C_A{}^{\dot{C}}a_{\dot{C}B}. \tag{8·1}$$

We may also form a skew-symmetric spinor c_{AB} from a_{AB} as follows. We first form the six-vector A_α according to

$$A_\alpha = -\tfrac{1}{4}\bar{a}_\alpha^{AB} a_{AB}, \tag{8·2}$$

and then make use of the correspondence between six-vectors and skew-symmetric spinors to define

$$c_{AB} = A_\alpha \gamma_{\alpha A}{}^{C} \epsilon_{CB}. \tag{8·3}$$

We can show that c_{AB} is in fact the skew-symmetric part of b_{AB}.

Proof. We have

$$\begin{aligned} A_\alpha &= -\tfrac{1}{4}\operatorname{tr}(a\tilde{\bar{a}}_\alpha) \\ &= -\tfrac{1}{4}\operatorname{tr}(a\tilde{\bar{\gamma}}_\alpha \tilde{B}^{-1}) \\ &= -\tfrac{1}{4}\operatorname{tr}(a\epsilon^{-1}\bar{\gamma}_\alpha \epsilon \tilde{B}^{-1}) \\ &= -\tfrac{1}{4}\operatorname{tr}(a\epsilon^{-1}\bar{\gamma}_\alpha C) \\ &= -\tfrac{1}{4}\operatorname{tr}(b\epsilon^{-1}\bar{\gamma}_\alpha). \end{aligned}$$

Now, all the $\epsilon^{-1}\bar{\gamma}_\alpha$ are skew-symmetric, so this expression involves only the skew-symmetric part of b, which we will denote by $\{b\}$. Being skew-symmetric, $\{b\}$ is a linear combination of the $\gamma_\alpha\epsilon$, so $\{b\}\epsilon^{-1}$ is a linear combination of γ_α

$$\{b\}\epsilon^{-1} = b_\alpha \gamma_\alpha.$$

Hence

$$A_\alpha = -\tfrac{1}{4}b_\beta \operatorname{tr}(\gamma_\beta \bar{\gamma}_\alpha) = b_\beta \delta_{\beta\alpha} = b_\alpha,$$

so that

$$\{b\}\epsilon^{-1} = A_\alpha \gamma_\alpha,$$

$$\{b\} = A_\alpha \gamma_\alpha \epsilon = c,$$

which is the required result.

9. *Spinor components in a specific representation.* In this section we shall employ a specific representation of the generators of the Dirac algebra, in order to write out the components of the spinor a_{AB} and c_{AB} of the previous section in terms of the six-vector A_α. We shall find relationships between spinor components in six-dimensions which are generalizations of certain spinor equations in a five-dimensional space which were first obtained by Eriksson (6) from quite different considerations. The theory already outlined in the present paper will have invested these expressions with a lucid geometrical and algebraic significance which they did not have in Eriksson's original work.

In a fixed reference frame, we will take for the γ_α the following matrices:

$$\sigma_1, \quad \sigma_2, \quad \sigma_3\rho_3, \quad \sigma_3\rho_2, \quad \sigma_3\rho_1, \quad i, \tag{9·1}$$

where the σ_i and ρ_i are two mutually commuting sets of (4×4) representations of C_2:

$$\left.\begin{aligned} \sigma_i &= S_i \otimes 1, \\ \rho_i &= 1 \otimes S_i \end{aligned}\right\} \quad (i = 1, 2, 3),$$

where S_i are the Pauli matrices given by (4·1). The first five of the γ_α are Hermitian, γ_6 skew-Hermitian, so in this reference frame B is a multiple of the unit matrix.

The matrix C must commute with those $\gamma_1, \dots \gamma_5$ that are real, and anticommute with those that are imaginary. Thus C is a multiple of $\sigma_1\rho_2$. We will take

$$C = -i\sigma_1\rho_2 = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix},$$

so that, following Eriksson in writing the spinor indices as $(1, 2, -2, -1)$, instead of $(1, 2, 3, 4)$, our expression

$$b_{AB} = C_A{}^{\dot{C}} a_{\dot{C}B} \tag{9·2}$$

becomes

$$a_{\dot{A}B} = (\operatorname{sgn} A)\, b_{-AB}.$$

The metric spinor is any multiple of C in our reference frame. We shall choose $\epsilon = -i\sigma_1\rho_2$, so that the matrices $\gamma_\alpha\epsilon$ become

$$-i\rho_2, \quad -\sigma_3\rho_2, \quad -i\sigma_2\rho_1, \quad \sigma_2, \quad i\sigma_2\rho_3, \quad -\sigma_1\rho_2.$$

The relation between a skew-symmetric spinor c_{AB} and a six-vector A_α,

$$c = A_\alpha\gamma_\alpha\epsilon$$

then takes the explicit form

$$c = \begin{pmatrix} \cdot & A_5 - iA_4 & -A_1 + iA_2 & -A_3 + iA_6 \\ -A_5 + iA_4 & \cdot & A_3 + iA_6 & -A_1 - iA_2 \\ A_1 - iA_2 & -A_3 - iA_6 & \cdot & -A_5 - iA_4 \\ A_3 - iA_6 & A_1 + iA_2 & A_5 + iA_4 & \cdot \end{pmatrix}. \tag{9·3}$$

The matrices $\bar{a}_\alpha$ are given by

$$-\sigma_1, \quad -\sigma_2, \quad -\sigma_3\rho_3, \quad -\sigma_3\rho_2, \quad -\sigma_3\rho_1, \quad -i.$$

The correspondence

$$A_\alpha = -\tfrac{1}{4}\bar{a}_\alpha^{\dot{A}B} a_{\dot{A}B}$$

can now be written out explicitly:

$$\left.\begin{aligned} 2(A_1 + iA_2) &= a_{\dot{1}2} + a_{-\dot{2}-1} = b_{-12} - b_{2-1} &&= 2c_{-12}, \\ 2(A_1 - iA_2) &= a_{\dot{2}1} + a_{-\dot{1}-2} = b_{-21} - b_{1-2} &&= 2c_{-21}, \\ 2(A_3 + iA_6) &= -a_{\dot{2}2} - a_{-\dot{2}-2} = b_{2-2} - b_{-22} &&= 2c_{2-2}, \\ 2(A_3 - iA_6) &= a_{\dot{1}1} + a_{-\dot{1}-1} = b_{-11} - b_{1-1} &&= 2c_{-11}, \\ 2(A_5 + iA_4) &= a_{\dot{1}-2} - a_{\dot{2}-1} = b_{-1-2} - b_{-2-1} &&= 2c_{-1-2}, \\ 2(A_5 - iA_4) &= a_{-\dot{2}1} - a_{-\dot{1}2} = b_{12} - b_{21} &&= 2c_{12}. \end{aligned}\right\} \tag{9·4}$$

Where c_{AB} is the skew-symmetric part of b_{AB}. We see immediately that the c given by these expressions coincides with the matrix (9·3) above, as we expected.

If we now consider only the subgroup of rotations about the fourth axis, A_4 becomes an invaraint, and the components $(A_1, A_2, A_3, iA_6, A_5)$ can be considered as a vector in a five-dimensional space with metric $(1, 1, 1, -1, 1)$, which was the space considered by Eriksson. The last two equations become

$$A_5 = \tfrac{1}{2}(a_{\dot{1}-2} - a_{\dot{2}-1} - a_{-\dot{1}2} + a_{-\dot{2}1}) = c_{12} + I$$

in five dimensions, where I is an invariant.

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REFERENCES

(1) BOERNER, H. *Representations of Groups* (North Holland Publishing Company, 1963).
(2) BRAUER, R. and WEYL, H. *Amer. J. Math.* **57** (1935), 425.
(3) DIRAC, P. A. M. *Proc. Roy. Soc. London Ser. A* **117** (1928), 350.
(4) EDDINGTON, SIR A. S. *Relativity Theory of Protons and Electrons* (Cambridge University Press, 1936).
(5) ELLIS, J. R. *Proc. Roy. Irish Acad. Sect. A* **64** (1966), 127.
(6) ERIKSSON, H. A. S. *Ark. Mat. Astr. Fys.* 29A **14**, (1943), 29.
(7) LORD, E. A. *Proc. Cambridge Philos. Soc.* **63** (1967), 785.
(8) PAULI, W. *Ann. Physik.* **18** (1933), 337.
(9) RASTALL, P. *Rev. Modern Phys.* **36** (no. 3) (1964), 820.
(10) ROMAN, P. *Theory of Elementary Particles* (North Holland Publishing Co. 1964).