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Clifford algebras in general relativity

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Summary. The generalization of the Dirac and quaternion algebras to Riemannian spaces is outlined. The components of various elements of the algebras are interpreted as physical quantities (tensors) and their symmetries and algebraic properties are linked with the properties of the algebra. The generalization of the quaternion algebra is of particular interest in that it resolves the anomalies that arise in the usual identification of quaternions with rank two spinors. Algebraic expressions for the electromagnetic energy momentum tensor, the Ricci tensor and Einstein tensor are obtained in both E-number and quaternion form. Extension of the principles to EF-numbers yields a proof of the symmetry properties of Bel's tensor and a simple expression for its divergence.

1. *Introduction.* The mathematical relationships that describe the properties of gravitational and electromagnetic fields find their most natural expression in terms of the calculus of tensors. The gravitational field is in fact a phenomenon arising out of differential geometric aspects of a four-dimensional Riemannian manifold with signature -2, while the electromagnetic field manifests itself as a rank two skew symmetric tensor on the manifold. The invariance group for gravitational and electromagnetic relationships is therefore the group of general coordinate transformations, and in this context the homogeneous Lorentz group has only very minor significance as the symmetry group of infinitesimal neighbourhoods. The existence of spinor fields, however, is evidence of a more intricate structure underlying physical law than can be dealt with on the simple hypothesis of space-time as a Riemannian manifold. The appropriate mathematical formulation of spinor relationships depends on the matrix-algebraic aspects of certain Clifford algebras, and the Lorentz group is the invariance group for spinor equations. Thus we have two conflicting aspects of modern physics; namely the differential aspects and matrix algebraic aspects. The two mathematical disciplines have been developed rapidly side by side, while much has remained bewildering, and even apparently inconsistent, about their relationship to each other. The present paper is an attempt to clarify the nature of this relationship, and at the same time to indicate a means of dealing with tensor relationships by matrix methods.

The defining relation for the generators e_a of a Clifford algebra C_n is the set of anticommutation rules

$$\tfrac{1}{2}(e_a e_b + e_b e_a) = -\delta_{ab} \quad (a, b = 1, \ldots, n)$$

on a set of n quantities e_a. The basis of the algebra consists of the 2^n linearly independent products that can be formed from the e_a. We denote the generators of C_2 (the quaternion algebra or Pauli algebra) by σ_1, σ_2, and the base elements by e_a $(a = 0 \ldots 3)$ with $\sigma_3 = -i\sigma_1\sigma_2$, $\sigma_0 = 1$. The Dirac algebra or E-number algebra is obtained from C_4 by a trivial modification of the above commutation rules—the Dirac matrices will be denoted by E_a $(a = 0 \ldots 3)$ and obey the relations

$$\tfrac{1}{2}(E_a E_b + E_b E_a) = \eta_{ab} = dg(1, -1, -1, -1).$$

The presence of the Minkowski metric on the right-hand side allows us to identify real components of a general element of the algebra with physical quantities, without the complication of 'reality conditions'.

The prominent position of the quaternion and Dirac algebras in special relativity is a simple consequence of the transformation law of an elementary spinor, whereby a general quaternion $p = p^a\sigma_a$ transforms as a rank two elementary spinor whenever p^a transforms as a Lorentz vector. Thus if $\lambda^a{}_b$ is the matrix of a Lorentz transformation, q the unimodular matrix of the corresponding spinor transformation, then

$$p^a \rightarrow \lambda^a{}_b p^b \quad \text{implies} \quad p \rightarrow qpq^\dagger.$$

Rastall extended this identification of vectors and quaternions to a Riemannian manifold (9). A similar, but more complex—and therefore more fruitful—situation holds for the Dirac algebra, where the components of a general E-number transform as the components of a scalar, a vector, a skewsymmetric rank two tensor, a pseudovector and a pseudoscalar under certain spinor transformations of the E-numbers. This fact was used extensively by Eddington (2, 3), and further investigated by Kilmister (5, 6).

The present paper demonstrates the possibilities inherent in extending this interpretation of E-number components to a Riemannian manifold, by using the Tetrode anticommutators $\tfrac{1}{2}(E_\mu E_\nu + E_\nu E_\mu) = g_{\mu\nu}$. Sections 2 and 3 deal with the algebraic and differential properties of this relation, without reference to physical application. In section 4 the results are used to formulate electromagnetism in an algebraic form, and an extremely simple expression for the electromagnetic energy-momentum tensor is obtained. The ideas concerning the Dirac algebra outlined in these Sections are then applied to the quaternion algebra, but to achieve full covariance under both types of transformation (change of coordinate system and Lorentz rotation of the vierbein), it is found necessary to introduce an arbitrary unit time-like vector into the geometry. This procedure is also found to clarify some of the anomalies that exist in the usual identification that can be made between the quaternion algebra and the algebra of rank two spinors. Section 7 deals with the Riemann, Ricci and Einstein tensors by means of the methods developed in the previous sections, and in the final section we extend the methods to the algebra formed from a direct product of two Dirac algebras—the 256 component algebra of EF-numbers, first introduced by Eddington (2, 3). An EF-number expression for Bel's tensor, analogous to the E-number formulation of the electromagnetic energy momentum obtained in section 4, enables the symmetry properties of Bel's tensor to be derived.

2. *Algebraic properties of the Dirac algebra.* (*a*) *Generalization of E-numbers to curvilinear coordinates.* The Dirac algebra was first introduced into physics in Dirac's flat-space equation for a fermion,

$$(E^a\partial_a - \kappa)\psi = 0,$$

where $\kappa = mc/\hbar$, $\partial_a = \partial/\partial x^a$ and the coordinates x^a $(a = 0 \ldots 3, x^0 = ct)$ are Cartesian coordinates. The description of such a particle in curvilinear coordinates and more generally in a Riemannian manifold, requires the introduction of vierbein fields—a set of orthonormal reference vectors $h^\mu{}_a$ ($a = 0 \ldots 3$ are labels that enumerate the vectors of the set, $\mu = 0 \ldots 3$ are coordinate indices of their components) at each point of the manifold. We suppose these vectors to vary continuously from point to point. The orthonormality implies

$$h^\mu{}_a h_{\mu b} = \eta_{ab},$$
$$h_{\mu a} h_\nu{}^a = g_{\mu\nu}.$$

Such reference fields enable the special theory of relativity to be incorporated into the general theory simply and naturally, and give an alternative description of gravitation, in some contexts more natural than metric components.

The matrices E^a (using η^{ab} to raise 'vierbein' indices) are regarded as constant and invariant under both coordinate transformations and under Lorentz rotations of the vierbein. They satisfy

$$\tfrac{1}{2}(E^a E^b + E^b E^a) = \eta^{ab}$$

and the complete Dirac algebra is given by the 16 base elements

$$1,\ E^a,\ E^{ab} = \tfrac{1}{2}(E^a E^b - E^b E^a), \quad E^5 = iE^0E^1E^2E^3 \quad \text{and} \quad E^{a5} = E^a E^5 = -E^5 E^a.$$

Abbreviating $E^aE^bE^c$ to E^{abc}, we have

$$E^{15} = iE^{230},$$
$$E^{25} = -iE^{130},$$
$$E^{35} = iE^{120},$$
$$E^{05} = iE^{123},$$

so that, in general,

$$E^{a5} = \frac{1}{3!} i\eta^{ab} \epsilon_{bcde} E^{cde}, \tag{2·1}$$

where ϵ_{bcde} is the usual 'alternating' symbol.

The quantities we wish to deal with are the $E^\mu = h^\mu{}_a E^a$ which satisfy

$$\tfrac{1}{2}(E^\mu E^\nu + E^\nu E^\mu) = g^{\mu\nu}.$$

Similarly,

$$E^{\mu\nu} = h^\mu{}_a h^\nu{}_b E^{ab} = \tfrac{1}{2}(E^\mu E^\nu - E^\nu E^\mu).$$

Defining $h = \det(h^\mu{}_a) = (\det h_\mu{}^a)^{-1}$, we have, from the well-known expression for a determinant,

$$\epsilon_{abcd} = h^{-1} \epsilon_{\mu\nu\rho\sigma} h^\mu{}_a h^\nu{}_b h^\rho{}_c h^\sigma{}_d \tag{2·2}$$

and taking the determinant of $h^\mu{}_a g_{\mu\nu} h^\nu{}_b = \eta_{ab}$ we have

$$h^{-1} = \pm(-g)^{\frac{1}{2}}, \quad \text{where} \quad g = \det(g_{\mu\nu}). \tag{2·3}$$

The sign, once chosen, is invariant under Lorentz rotations of the vierbein and coordinate transformations, provided neither type of transformation involves a reflexion. We choose the positive sign. It is then possible by a continuous coordinate transformation to obtain locally Cartesian coordinates for which the $h^\mu{}_a$ are oriented along the positive coordinate directions. Substituting (2·2) and (2·3) in (2·1) we have

$$E_{\mu 5} = h_{\mu a}E^{a5} = -(1/3!)\,g^{\frac{1}{2}}\epsilon_{\lambda\nu\rho\sigma}E^{\nu\rho\sigma}.$$

Now g is a scalar density of weight 2 and the Levi–Civita tensor densities $\epsilon_{\lambda\sigma\nu\rho}$, $\epsilon^{\lambda\nu\rho\sigma}$ have weights -1 and $+1$ respectively. We therefore define the *alternating pseudo-tensors*

$$e_{\lambda\nu\rho\sigma} = g^{\frac{1}{2}}\epsilon_{\lambda\nu\rho\sigma},$$
$$e^{\lambda\nu\rho\sigma} = g^{-\frac{1}{2}}\epsilon^{\lambda\nu\rho\sigma}.‡$$

It is then easily seen that they are the covariant and contravariant forms of the *same* tensor, since

$$-g_{\alpha\mu}g_{\beta\nu}g_{\gamma\rho}g_{\delta\sigma}\epsilon^{\mu\nu\rho\sigma} = g\epsilon_{\alpha\beta\gamma\delta}.$$

The expression for $E_{\mu 5}$ is now simply

$$E_{\mu 5} = -(1/3!)\,e_{\mu\nu\rho\sigma}E^{\nu\rho\sigma}, \tag{2·4}$$

and since $E_\mu E^\mu = 4$,

$$E_5 = -(1/4!)\,e_{\mu\nu\rho\sigma}E^{\mu\nu\rho\sigma}. \tag{2·5}$$

We shall require an expression for the components of the elements $E^{\alpha\beta\gamma}$ of the Dirac algebra. From (2·4) we obtain

$$e^{\mu\alpha\beta\gamma}E_{\mu 5} = -(1/3!)\,e^{\mu\alpha\beta\gamma}e_{\mu\nu\rho\sigma}E^{\nu\rho\sigma}.$$

Now,

$$e^{\mu\alpha\beta\gamma}e_{\mu\nu\rho\sigma} = 3!\,\delta^{\alpha\beta\gamma}_{[\nu\rho\sigma]},$$

where $\delta^{\alpha\beta\gamma}_{\nu\rho\sigma} = \delta^\alpha_\nu\delta^\beta_\rho\delta^\gamma_\sigma$ and the square bracket denotes complete antisymmetrization

$$a_{[\nu\rho\sigma]} = 1/3!(a_{\nu\rho\sigma}+a_{\rho\sigma\nu}+a_{\sigma\nu\rho}-a_{\nu\sigma\rho}-a_{\sigma\rho\nu}-a_{\rho\nu\sigma}).$$

Hence

$$e^{\mu\alpha\beta\gamma}E_{\mu 5} = -E^{[\alpha\beta\gamma]} = 1/3!\{-6E^{\alpha\beta\gamma}+6(g^{\gamma\beta}E^\alpha - g^{\alpha\gamma}E^\beta + g^{\alpha\beta}E^\gamma)\},$$

where we have made use of the anticommutation rules. We finally obtain

$$E^{\alpha\beta\gamma} = e^{\alpha\beta\gamma\mu}E_{\mu 5}+g^{\alpha\beta}E^\gamma - g^{\gamma\alpha}E^\beta + g^{\beta\gamma}E^\alpha. \tag{2·6}$$

(*b*) *Identification of components with vectors and tensors.* We require that a general E-number

$$P = p+p_\mu E^\mu + \tfrac{1}{2}p_{\mu\nu}E^{\mu\nu}+p_{\mu 5}E^{\mu 5}+p_5E^5$$

shall be invariant under coordinate transformations, and that under a Lorentz rotation $\lambda_a{}^b$ of the vierbeins ($h^\mu{}_a = \lambda_a{}^b h^\mu{}_b$) it shall transform to QPQ^{-1}, where Q is given by

$$E_a = \lambda_a{}^b QE_bQ^{-1}.$$

This ensures that p, p_μ, $p_{\mu\nu}$, $p_{\mu 5}$ and p_5 are respectively the components of a scalar, a vector, a skewsymmetric rank two tensor, a pseudovector and a pseudoscalar. The most important E-numbers for our purposes are those of the form $x = x_\mu E^\mu$ and $f = \frac{1}{2}f_{\mu\nu}E^{\mu\nu}$. The scalar product of two vectors is then given by

$$g_{\mu\nu}x^\mu y^\nu = x_\mu y^\mu = \tfrac{1}{2}(xy+yx), \tag{2·7}$$

where

$$x = x_\mu E^\mu, \quad y = y_\mu E^\mu.$$

‡ More explicitly $i\sqrt{(-g)}\,\epsilon_{\lambda\nu\rho\sigma}$ and $1/i\sqrt{(-g)}\,\epsilon^{\lambda\nu\rho\sigma}$, with the positive square root.

The components x_μ are obtained from x by

$$x_\mu = \tfrac{1}{2}(E_\mu x + xE_\mu) \tag{2·8}$$

or, since all the base elements of the algebra except the unit matrix are traceless,

$$x_\mu = \tfrac{1}{4}\,\mathrm{tr}\,(E_\mu x). \tag{2·9}$$

The quantity $x_\mu f^{\mu\nu}$ is given by

$$x_\mu f^{\mu\nu} E_\nu = \tfrac{1}{2}(xf - fx). \tag{2·10}$$

Equations (2·7), (2·8) and (2·10) are all simple consequences of the anticommutation rules.

(*c*) *Duals of skewsymmetric tensors.* The dual of $f^{\mu\nu}$ is defined to be

$$f^{*\mu\nu} = \tfrac{1}{2}\, e^{\mu\nu\rho\sigma} f_{\rho\sigma}.$$

If we define the covariant dual $f^*_{\mu\nu} = \tfrac{1}{2} e_{\mu\nu\rho\sigma} f^{\rho\sigma}$, then the operation of taking the dual clearly commutes with that of raising and lowering indices. Defining the E-number $E^{*\mu\nu}$ by

$$E^{*\mu\nu} = \tfrac{1}{2} e^{\mu\nu\rho\sigma} E_{\rho\sigma}$$

we have, for $f^\mu = \tfrac{1}{2} f_{\mu\nu} E^{\mu\nu}$,

$$\tfrac{1}{2} f^*_{\mu\nu} E^{\mu\nu} = \tfrac{1}{2} f_{\mu\nu} E^{*\mu\nu} = f^*$$

and

$$\begin{aligned} E^5 E^{*\rho\sigma} &= \tfrac{1}{2} E_5 e^{\rho\sigma\mu\nu} E_{\mu\nu} \\ &= \tfrac{1}{2} E_\mu (E^{\rho\sigma\mu} - g^{\rho\sigma} E^\mu + g^{\mu\rho} E^\sigma - g^{\sigma\mu} E^\rho) \\ &= E^{\rho\sigma} \end{aligned}$$

(where we have made use of (3·6) and the identify $E_\mu E^{\rho\sigma} E^\mu = 0$). Thus, in the E-number algebra, the dual of a skewsymmetric tensor is given simply by

$$f^* = E_5 f = f E_5. \tag{2·11}$$

(*d*) *Expressions for certain components of a general E-number.* A general E-number P can be written

$$P = p + q + r + s + t,$$

where $\quad p = \tfrac{1}{4}\,\mathrm{tr}\,P,\ q = p_\mu E^\mu,\ r = \tfrac{1}{2} p_{\mu\nu} E^{\mu\nu},\ S = p_{\mu 5} E^{\mu 5},\ t = p_5 E^5.$

We have $\quad p_\mu = \tfrac{1}{4}\,\mathrm{tr}\, E_\mu P, \quad p_{\mu\nu} = \tfrac{1}{4}\,\mathrm{tr}\, E_{\mu\nu} P,$

etc. (see Eddington (2)). Now, for a 'pure vector' E-number (i.e. $P = q$) we found an alternative expression for the components (2·8), namely $p_\mu = \tfrac{1}{2}(E_\mu P + PE_\mu)$. We might expect this to be a special case of more general expressions for components of E-numbers. To obtain such expressions, we evaluate the quantities

$$Q = E^\mu P E_\mu, \quad R = \tfrac{1}{2} E^{\mu\nu} P E_{\mu\nu}, \quad S = E^{\mu 5} P E_{\mu 5}, \quad T = E^5 P E_5,$$

and obtain

$$\left.\begin{aligned} P &= p + q + r + s + t, \\ Q &= 4p - 2q + 2s - 4t, \\ R &= -6p + 2r - 6t, \\ S &= -4p - 2q + 2s - 4t, \\ T &= p - q + r - s + t. \end{aligned}\right\} \tag{2·12}$$

Solving for p by Cramer's rule

$$p = \tfrac{1}{4}\operatorname{tr} P = \tfrac{1}{16}(P+Q-R-S+T). \tag{2·13}‡$$

Using (2·13) on the E-number $E_\mu P$ instead of P, we find

$$\begin{aligned}\tfrac{1}{4}\operatorname{tr} E_\mu P &= \tfrac{1}{16}(E_\mu P + E^\rho E_\mu P E_\rho - \tfrac{1}{2}E^{\rho\sigma}E_\mu P E_{\rho\sigma} - E^{\rho 5}E_\mu P E_{\rho 5} + E^5 E_\mu P E_5)\\ &= \tfrac{1}{16}\{E_\mu(3P-Q-R-S-T) + (2P-2Q-2T)\,E_\mu\},\end{aligned}$$

where use has been made of the anticommutation rules. The same quantity can be expressed as

$$\tfrac{1}{4}\operatorname{tr} PE_\mu = \tfrac{1}{16}\{(3P-Q-R-S-T)\,E_\mu + E_\mu(2P-2Q-2T)\}.$$

Adding,

$$p_\mu = \tfrac{1}{32}\{E_\mu(5P-3Q-R-S-3T) + (5P-3Q-R-S-3T)\,E_\mu\}, \tag{2·14}$$

which gives a means of picking out the vector components of a general E-number, which does not depend on the expression $p_\mu = \tfrac{1}{4}\operatorname{tr} E_\mu P$. Note that $(5P-3Q-R-S-3T)$ is $16q$ as can be seen by solving (2·12) for q using Cramer's rule. For a pure vector quantity $P = q$ and the expression (2·14) reduces to (2·8).

In the same way, we may derive an expression for $p_{\mu\nu}$—the tensor components of a general E-number. Using (2·13) with $E_{\mu\nu}P$ instead of P,

$$\begin{aligned}p_{\mu\nu} = \tfrac{1}{4}\operatorname{tr} PE_{\mu\nu} = \tfrac{1}{16}\{&(P+Q-R-S+T)\,E_{\mu\nu} + 2E_\nu(P+Q+T)\,E_\mu\\ &+ 2E_\mu(P+Q+T)\,E_\nu + 4(E_{\mu\nu}P + PE_{\mu\nu})\}.\end{aligned}$$

Expressing the same quantity as $\tfrac{1}{4}\operatorname{tr} E_{\mu\nu}P$ changes only the first term which becomes $E_{\mu\nu}(P+Q-R-S+T)$. Adding the two expressions gives an expression for the tensor part of any E-number

$$\begin{aligned}p_{\mu\nu} = \tfrac{1}{32}\{&E_{\mu\nu}(9P+Q-R-S+T) + (9P+Q-R-S+T)\,E_{\mu\nu}\}\\ &- \tfrac{1}{8}\{E_\mu(P+Q+T)\,E_\nu - E_\nu(P+Q+T)\,E_\mu\}.\end{aligned} \tag{2·15}$$

For a skewsymmetric tensor quantity $P = \tfrac{1}{2}p_{\mu\nu}E^{\mu\nu} = r$ we have

$$9P+Q-R-S+T = 8P,$$

$$P+Q+T = 2P,$$

and the formula reduces to

$$p_{\mu\nu} = \tfrac{1}{4}(E_{\mu\nu}P + PE_{\mu\nu} - E_\mu PE_\nu + E_\nu PE_\mu), \tag{2·16}$$

which is an expression for the components of an E-number containing only a tensor part, completely analogous to (2·8) for an E-number with only a vector part.

3. *Differential properties.* (*a*) *The differential operators.* So far we have investigated only the properties of the E-number algebra at a point; we now turn our attention to the differential calculus that arises from a field of such quantities. We have three distinct differential operators:

(i) $\partial_\mu = \partial/\partial x^\mu$, the ordinary partial derivative, which may be combined with the E-number to form an operator $E^\mu\partial_\mu = \partial$ operating to the right (or $\partial_\mu E^\mu$ operating to the left).

‡ Eddington's E-numbers differ from those used here by a factor $\pm i$ which would give $p = \tfrac{1}{16}(P+Q+R+S+T)$.

(ii) For covariant differentiation, we may fix the vierbein so that the requirement of invariance under Lorentz rotation is relaxed. Vierbein indices and the spinor indices labelling the rows and columns of the E-numbers are not subject to transformation laws. The appropriate covariant differentiation is then denoted by D_μ. Then

$$D_\mu E_\sigma = (D_\mu h_\sigma{}^a) E_a = (\partial_\mu h_\sigma{}^a - \Gamma^\rho{}_{\mu\sigma} h^a{}_\rho) E_a$$
$$= E_a \gamma^a{}_{\sigma\mu} = E^\rho \gamma_{\rho\sigma\mu}, \tag{3·1}$$

which may be taken as the *definition* of the spin coefficients $\gamma_{\rho\sigma\mu}$. Their skewsymmetry in $(\rho\sigma)$ then follows from

$$0 = D_\mu g_{\rho\sigma} = \tfrac{1}{2} D_\mu (E_\rho E_\sigma + E_\sigma E_\rho) = \gamma_{\sigma\rho\mu} + \gamma_{\rho\sigma\mu}.$$

(iii) The complete covariant derivative which takes account of vierbein and spinor indices as well as coordinate indices:

$$\nabla_\mu \psi = \partial_\mu \psi - \Gamma_\mu \psi,$$
$$\nabla_\mu \phi = \partial_\mu \phi + \phi \Gamma_\mu,$$

where ψ and ϕ are a covariant and a contravariant four-component spinor, respectively. The Γ_μ are a set of E-numbers (the spinor connexion). The general covariance of the Dirac equation $(E^\mu \nabla_\mu - \kappa)\psi = 0$ then requires that

$$\nabla_\mu E_\sigma = D_\mu E_\sigma + E_\sigma \Gamma_\mu - \Gamma_\mu E_\sigma. \tag{3·2}$$

The operator $E^\mu \nabla_\mu$, operating to the right or left, will be denoted by ∇.

The quantities Γ_μ are not uniquely defined—we only require their transformation law to be such that $(\partial_\mu - \Gamma_\mu)\psi$ is a vector under coordinate transformations, and a spinor under Lorentz transformations. We can fix Γ_μ by the condition that

$$E_{\sigma;\mu} = \nabla_\mu E_\sigma = 0, \tag{3·3}$$

which is analogous to the requirement $g_{\mu\nu:\sigma} = 0$ to fix the components of the affine connexion. We then find

$$E^\rho \gamma_{\rho\sigma\mu} = -E_\sigma \Gamma_\mu + \Gamma_\mu E_\sigma, \tag{3·4}$$

so

$$\gamma_{\rho\sigma\mu} E^{\rho\sigma} = -E_\sigma \Gamma_\mu E^\sigma + 4\Gamma_\mu. \tag{3·5}$$

Writing

$$\Gamma_\mu = a_\mu + a_{\mu\sigma} E^\sigma + \tfrac{1}{2} a_{\mu\rho\sigma} E^{\rho\sigma} + a_{\mu\sigma 5} E^{\sigma 5} + a_{\mu 5} E^5$$

for the components of the spinor connexion and using the relations

$$E_\sigma E^\sigma = 4, \quad E_\sigma E^\mu E^\sigma = -2E^\mu, \quad E_\sigma E^{\mu\nu} E^\sigma = 0,$$
$$E_\sigma E^{\mu 5} E^\sigma = 2E^{\mu 5}, \quad E_\sigma E^5 E^\sigma = -4E^5,$$

equation (3·5) implies that $2a_{\rho\mu\nu} = \gamma_{\mu\nu\rho}$, a_μ can be chosen arbitrarily, and the rest of the components are zero.

Thus

$$\Gamma_\mu = \tfrac{1}{4} \gamma_{\rho\sigma\mu} E^{\rho\sigma} + a_\mu. \tag{3·6}$$

A 'curvature spinor' $M_{\mu\nu}$ is a spin-tensor obtained by operating on a covariant spinor with the commutator of two complete covariant derivatives, i.e.

$$(\nabla_{\mu\nu} - \nabla_{\nu\mu})\psi = M_{\mu\nu}\psi.$$

We obtain

$$M_{\mu\nu} = \partial_\nu \Gamma'_\mu - \partial_\nu \Gamma'_\nu + \Gamma'_\mu \Gamma'_\nu - \Gamma'_\nu \Gamma'_\mu + f_{\mu\nu}, \tag{3·7}$$

where $$\Gamma'_\mu = \tfrac{1}{4}\gamma_{\rho\sigma\mu}E^{\rho\sigma} \quad \text{and} \quad f_{\mu\nu} = \partial_\nu a_\mu - \partial_\mu a_\nu.$$

The spin coefficients vanish in a flat space with Cartesian coordinates and a natural vierbein, so that the covariant derivative operator for a covariant spinor in this case becomes $(\partial_\mu - a_\mu)$, which is just the modification that the operator ∂_μ of Dirac's equation undergoes when an electromagnetic field is present, if a_μ is interpreted as $ieA_\mu/\hbar c$, with A_μ the electromagnetic potential. On this basis it is possible to identify

$$f_{\mu\nu} = \tfrac{1}{4}\operatorname{tr} M_{\mu\nu}$$

of (3·7) with the Maxwell tensor.

4. *Electromagnetism.* The Maxwell equations can be written

$$\left.\begin{aligned} Sf_{\mu\rho;\sigma} &= 0, \\ f^{\rho\mu}{}_{;\rho} &= j^\mu, \end{aligned}\right\} \tag{4·1}$$

where S indicates summation of the three terms obtained by cyclically permuting $(\mu\rho\sigma)$;

$$Sf_{\mu\rho;\sigma} = f_{\mu\rho;\sigma} + f_{\rho\sigma;\mu} + f_{\sigma\mu;\rho}.$$

Now $Sf_{\mu\rho;\sigma} = 0$ is equivalent to $\quad f^{*\rho\mu}{}_{;\mu} = 0,$

so making use of (2·10) these can be written as

$$\tfrac{1}{2}(\nabla f^* - f^*\nabla) = 0,$$
$$\tfrac{1}{2}(\nabla f - f\nabla) = j,$$

where $j = j_\mu E^\mu$. Since $f^* = E^5 f$, the first of these equations is equivalent to $\tfrac{1}{2}(\nabla f + f\nabla) = 0$ so in our formalism the equations (4·1) are

$$\nabla f = j = -f\nabla. \tag{4·2}$$

The electromagnetic energy-momentum tensor can be defined as the quantity $T^{\mu\nu}$ whose divergence is the Lorentz force

$$T^{\mu\nu}{}_{;\mu} = -f^{\sigma\nu}j_\sigma.$$

In our matrix formalism, this is

$$\begin{aligned}(T^{\mu\nu}E_\nu)_{;\mu} &= f^{\sigma\nu}j_\nu E^\sigma \\ &= \tfrac{1}{2}(fj - jf),\end{aligned}$$

where we have used (2·10). But making use of the Maxwell equations in the form (4·2) this is

$$\tfrac{1}{2}(f(\nabla f) + (f\nabla)f) = \tfrac{1}{2}(fE^\mu f_{;\mu} + f_{;\mu}E^\mu f) = (\tfrac{1}{2}fE^\mu f)_{;\mu}.$$

Thus the energy momentum tensor may be defined by

$$T^{\mu\nu}E_\nu = \tfrac{1}{2}fE^\mu f. \tag{4·3}$$

Its symmetry and the vanishing of its trace can be obtained at once by multiplication by E_μ, making use of the identity $E_\mu E^{\rho\sigma}E^\mu = 0$:

$$\tfrac{1}{2}(T^{\mu\nu} - T^{\nu\mu})E_{\mu\nu} + T^\sigma{}_\sigma = E_\mu E_\nu T^{\mu\nu} = \tfrac{1}{2}E_\mu f E^\mu f = \tfrac{1}{4}f_{\rho\sigma}E_\mu E^{\rho\sigma}E^\mu f = 0,$$

hence $$T^\sigma{}_\sigma = 0, \quad T^{\mu\nu} = T^{\nu\mu}. \tag{4·4}$$

The usual expression for $T^{\mu\nu}$ can be obtained by repeated use of (2·6) to evaluate the components of $fE_\mu f$. It is simpler to obtain a matrix formulation for

$$f_{\mu\sigma}f_\nu{}^\sigma - \tfrac{1}{4}g_{\mu\nu}f_{\alpha\beta}f^{\alpha\beta} \tag{4·5}$$

and hence to show that this is the above tensor. (4·5) can be rewritten as

$$\tfrac{1}{2}(f_{\mu\sigma}f_\nu{}^\sigma - f^*{}_{\mu\sigma}f^*{}_\nu{}^\sigma),$$

so we require an expression for $f_{\mu\sigma}f_\nu{}^\sigma$ where $f_{\mu\nu}$ is a skewsymmetric tensor. Writing $f = \frac{1}{2}f_{\mu\nu}E^{\mu\nu}$ we have

$$\tfrac{1}{2}(E_\alpha f - fE_\alpha) = f_{\alpha\nu}E^\nu = f_\alpha.$$

Using (2·10) with f_μ for x and $k_{\mu\nu}$ for $f_{\mu\nu}$ we have

$$f_{\mu\nu}k^{\nu\sigma}E_\sigma = \tfrac{1}{2}(f_\mu k - kf_\mu) = \tfrac{1}{4}(E_\mu fk - kE_\mu f - fE_\mu k + kfE_\mu). \tag{4·6}$$

Thus

$$f_{\nu\mu}f^{\nu\mu}E_\sigma = \tfrac{1}{4}(2fE_\mu f - f^2E_\mu - E_\mu f^2) \tag{4·7}$$

and $f^*{}_{\nu\mu}f^{*\nu\sigma}E_\sigma$ is the same quantity with f replaced by E_5f or fE_5. Subtracting the two expressions,

$$\tfrac{1}{2}(f_{\mu\sigma}f_\nu{}^\sigma - f^*{}_{\mu\sigma}f^*{}_\nu{}^\sigma)\,E_\sigma = \tfrac{1}{2}fE_\mu f, \tag{4·8}$$

which completes the proof that $\frac{1}{2}fE_\mu f$ does in fact yield the usual energy momentum tensor;

$$T_{\mu\nu} = \tfrac{1}{8}\,\mathrm{tr}\,E_\mu fE_\nu f.$$

In section 8 an extension of the method enables us to express Bel's tensor in terms of EF-numbers, and thence obtain a simple proof of the symmetries of Bel's tensor.

5. *Generalization of the quaternion algebra.* (*a*) *Algebraic properties.* Before we can make use of the quaternion algebra, we must resolve the inconsistencies involved in the usual identification of quaternions with rank two elementary spinors (see for instance, Ellis (4)).

All four of the Dirac matrices anticommute, so that the four vierbein indices of E^1, E^2, E^3, E^0 are on an equal footing—consequently (defining $E^\mu = h^\mu{}_a E^a$), the expression for any *product* of E-numbers in terms of the basis is completely covariant, for Lorentz rotations of the vierbein as well as for coordinate transformations, e.g.

$$E^{\mu\nu\sigma} = e^{\mu\nu\sigma\rho}E_{\rho 5} + g^{\mu\nu}E^\sigma - g^{\sigma\mu}E^\nu + g^{\nu\sigma}E^\mu.$$

The situation for the quaternion algebra is rather more complicated, for σ_1, σ_2 and σ_3 anticommute, while $\sigma_0 = 1$ commute with the other three. Thus time is singled out from the three space directions, so we cannot expect invariance of quaternion expressions under Lorentz rotations that involved the time-like vierbein $h^\mu{}_0$. The situation is further clarified by considering the actual transformation laws. Under a Lorentz rotation the E-numbers undergo a transformation of the form

$$E^\mu \to QE^\mu Q^{-1},$$

where Q is determined by the requirement of invariance of the E^a. Thus any *product* of E numbers has the same transformation law

$$E^{\mu\sigma\nu} \to QE^{\mu\nu\sigma}Q^{-1}.$$

The quaternions, however, transform according to

$$\sigma^\mu \to q\sigma^\mu q^\dagger,$$

where $q^\dagger$ is the Hermitian conjugate of q (i.e. as a rank two spinor with one 'undotted' covariant index and one 'dotted' covariant index), where q is a *unimodular* matrix determined so that the σ^a are unchanged. Thus, unless q is also *unitary*, the *products* of σ-matrices do not have a sensible transformation law. q is in fact unitary only for the spatial rotations—those not involving the fourth vierbein $h^\mu{}_0$.

The quaternion conjugate to $p = p_\mu \sigma^\mu$ is usually defined to be that formed by the inversion of the vector p_μ in the 3-space determined by $h^\mu h^\mu_k$ $(k = 1, 2, 3)$, i.e.

$$\bar{\sigma}_k = -\sigma_k \quad (k = 1, 2, 3),$$
$$\bar{\sigma}_0 = \sigma_0.$$

This definition of $\bar{p}$ is therefore dependent on the choice of vierbein, so that $\bar{p}$ is not the same kind of quantity as p. It has a different transformation law. Usually the transformation matrix q is considered as a quaternion, its unimodularity is expressed by $q\bar{q} = 1$ and the transformation law of $\bar{p}$ is then

$$\bar{p} \to \bar{q}^\dagger \bar{p} \bar{q}, \tag{5·1}$$

so that products of the form $\sigma^\mu \bar{\sigma}^\nu$ can be considered, as they possess a spinor transformation law

$$\sigma^\mu \bar{\sigma}^\nu \to q\sigma^\mu \bar{\sigma}^\nu \bar{q}.$$

However, this interpretation is not valid, since q is a spinor transformation matrix and therefore has two *undotted* indices, so that it cannot properly be expressed as a linear combination of the σ^μ. The quantity $\bar{q}$ can therefore not be defined in a covariant way.

The way out of these difficulties is clearly to convert the 'dotted' index of the elements σ^μ to an undotted index by multiplication on the right with a matrix $\bar{c}$ whose transformation law is

$$\bar{c} \to q^{\dagger -1} \bar{c} q^{-1}.$$

Thus we define

$$S^\mu = \sigma^\mu \bar{c}. \tag{5·2}$$

Then $S^\mu \to qS^\mu q^{-1}$ and products of the S-numbers (which we now call quaternions) can be formed. We define the 'special' vierbeins to be those in which $\bar{c}$ is the unit matrix. In these frames the S numbers are the usual quaternions and the structure constants defined by

$$S^\mu S^\nu = c^{\mu\rho\nu} S_\rho \tag{5·3}$$

are the usual structure constants of the quaternion algebra. In any other frame $\bar{c} = s^\dagger s$, where s is the matrix of the transformation to a special frame. The special frames are therefore mapped into each other by the unitary unimodular rotations—i.e. by 'spatial' rotations, and therefore a chosen set of special frames is characterized by a specified time direction at each point, or by an arbitrary chosen unit time-like vector field c^μ. This is the $h^\mu{}_0$ of the special frames, so that $c_\mu = (1, 0, 0, 0)$ in these frames and $c_\mu \sigma^\mu$ is the unit matrix. In any other frame related to the special ones by a transformation matrix s, $C = s^{-1}s^{\dagger -1}$ so that $\bar{c} = c^{-1}$ in any frame. Given a quaternion $p = p_\mu S^\mu$ we may define the conjugate to be the quaternion obtained by space inversion of p_μ *in the special frames*. Thus

$$\left.\begin{aligned} \bar{p}_\mu &= -p_\mu + 2p_\mu p_\nu c^\nu, \\ \text{or} \qquad \bar{S}_\mu &= -S_\mu + 2c_\mu, \end{aligned}\right\} \tag{5·4}$$

where we have used $S_\nu c^\nu = \sigma_\nu c^\nu = c\bar{c} = 1$. We now have a definition of quaternionic conjugation that is independent of the choice of vierbein and moreover the transformation law of $\bar{S}_\mu$ is the same as that of S_μ. It is convenient also to define the conjugates of the Pauli matrices σ_μ by the relation

$$\bar{S}_\mu = c\bar{\sigma}_\mu. \tag{5·5}$$

In the special frames this yields the usual definition of the σ_μ. Since a transformation matrix q can be written as an S-number $q_\mu S^\mu$ we have

$$\begin{aligned} q\bar{q} &= q_\mu q_\nu S^\mu \bar{S}^\nu = \tfrac{1}{2}q_\mu q_\nu (S^\mu \bar{S}^\nu + S^\nu \bar{S}^\mu) \\ &= q_\mu q_\nu g^{\mu\nu} = q_\mu q^\mu, \end{aligned}$$

and evaluating the determinant of q in the special frames we find $q_\mu q^\mu = |q| = 1$. Thus $\bar{q}$ is the inverse of a unimodular q and the transformation law for the $\bar{\sigma}_\mu$ deduced from (5·5) can be written

$$\bar{\sigma}_\mu \rightarrow \bar{q}^\dagger \bar{\sigma}_\mu \bar{q}.$$

Thus the $\bar{\sigma}_a$ are the usual $\tilde{\sigma}_a$, representing reflexions relative to the particular vierbein, for they take the same constant values for all frames:

$$\bar{\sigma}_a \rightarrow \lambda_a{}^b \bar{q}^\dagger \bar{\sigma}_b \bar{q} = \lambda_a{}^b \overline{q\sigma_b q^\dagger} = \bar{\sigma}_a.$$

We may evaluate the completely covariant structure constants defined by

$$S_\mu S_\nu = c_{\mu\rho\nu} S^\rho.$$

The $c_{\mu\rho\nu}$ are evaluated in the special frames from the usual multiplication rules of the Pauli matrices

$$\begin{aligned} &S_i S_j = i\epsilon_{ijk} S^k + \delta_{ij} = -S_j S_i \quad (i, j = 1, 2, 3), \\ &S_0 S_i = S_i = S_i S_0, \\ &S_0^2 = 1. \end{aligned}$$

These formulae can be combined in the single expression

$$S^a S^b = i\epsilon^{abc0} S_c + \delta^{0a} S^b + \delta^{0b} S^a - \eta^{ab} S^0 \quad (a, b = 0 \ldots 3),$$

i.e.

$$S^a S^b = e^{abcd} S_c c_d + c^a S^b + c^b S^a - \eta^{ab} S^0,$$

or, in coordinate indices,

$$S^\mu S^\nu = e^{\mu\rho\nu\sigma} S_\rho c_\sigma + c^\mu S^\nu + c^\nu S^\mu - g^{\mu\nu} c_\rho S^\rho.$$

Hence

$$c^{\mu\rho\nu} = e^{\mu\rho\nu\sigma} c_\sigma + g^{\mu\rho} c^\nu - g^{\nu\mu} c^\rho + g^{\rho\nu} c^\mu, \tag{5·6}$$

which is a fully covariant expression, true for all frames (except of course under reflexions of either vierbein or coordinate systems, for which $e^{\mu\rho\nu\sigma}$ undergoes a change of sign).

We may also evaluate the structure constants for the multiplication $S^{\mu}\bar{S}^{\nu}$ as follows:

$$S^{\mu}\bar{S}^{\nu} = -c^{\mu\rho\nu}S_{\rho} + 2c^{\nu}S^{\mu} = (-c^{\mu\rho\nu} + 2g^{\mu\rho}c^{\nu})\,S_{\rho} = c^{\rho\mu\nu}S_{\rho}.$$

We will also require the quaternions $S^{\mu\nu} = \frac{1}{2}(S^{\mu}\bar{S}^{\nu} - S^{\nu}\bar{S}^{\mu})$ which are in a sense analogous to the $E^{\mu\nu}$ of the Dirac algebra, except that they can be expressed as a linear combination of the generators S_{ρ}:

$$\begin{aligned} S^{\mu\nu} &= s^{\rho\mu\nu}S_{\rho} \\ &= \tfrac{1}{2}(c^{\rho\mu\nu} - c^{\rho\nu\mu})\,S_{\rho} \\ &= e^{\rho\mu\nu\sigma}c_{\sigma}S^{\rho} + S^{\mu}c^{\nu} - S^{\nu}c^{\mu} \\ &= (e^{\mu\nu\rho\sigma} + g^{\mu\nu\rho\sigma})\,S^{\rho}c_{\sigma}, \end{aligned} \tag{5·7}$$

where $g_{\mu\nu\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}$, which is the *dual* of $e_{\mu\nu\rho\sigma}$ over one of its index pairs. Incidentally we note that $g_{\mu\nu\rho\sigma}$ has the same algebraic symmetries as a Riemann tensor and its trace is zero. Any quaternion of the form

$$K = \tfrac{1}{2}f_{\mu\nu}S^{\mu\nu}$$

can be written

$$K = (f_{\mu\nu} + f^{*}_{\mu\nu})\,S^{\mu}c^{\nu}. \tag{5·8}$$

A similar expression involving the antiself-dual part of $f_{\mu\nu}$ is obtained by noting that the S_{μ} are Hermitian in the special frames (where they are just linear combinations of the Pauli matrices, with real coefficients) thus in general, in all frames,

$$cS^{\dagger}_{\mu}\bar{c} = S_{\mu}. \tag{5·9}$$

If f_{μ} is real, $f^{*}_{\mu\nu}$ is imaginary so combining the expression (5·8) with (5·9) we obtain

$$cK^{\dagger}\bar{c} = (f_{\mu\nu} - f^{*}_{\mu\nu})\,S^{\mu}c^{\nu}. \tag{5·10}$$

From the expression (5·7) for the components of $S_{\mu\nu}$ it is apparent that $S_{\mu\nu}$ is a 'pure' quaternion, that is,

$$S_{\mu\nu} + \bar{S}_{\mu\nu} = 0.$$

Thus every quaternion of the form $K = \frac{1}{2}f_{\mu\nu}S^{\mu\nu}$ is 'pure'. We also have the *converse*, that every *pure* quaternion $K = K_{\mu}S^{\mu}$ can be written in the form $K = \frac{1}{2}f_{\mu\nu}S^{\mu\nu}$. For, if K is pure we have $K + \bar{K} = 0$ which gives $K_{\mu}c^{\mu} = 0$, so

$$\begin{aligned} K_{\mu}c_{\nu}S^{\mu\nu} &= K_{\mu}c_{\nu}c_{\sigma}g^{\mu\nu\rho\sigma}S_{\rho} \\ &= K^{\rho}S_{\rho} - K^{\rho}c_{\rho} = K. \end{aligned}$$

Hence K has the form $\frac{1}{2}f_{\mu\nu}S^{\mu\nu}$, where $f_{\mu\nu} = K_{\mu}c_{\nu} - c_{\nu}K_{\mu}$.

(*b*) *Differential properties.* Having modified the concept of quaternions so as to eliminate the anomalies introduced by the usual identification of quaternions with rank two spinors, and investigated the algebraic properties of the quantities S_{μ}, we are now able to deal with the differential properties. These will involve the differential properties of the vector field c_{μ} that we have introduced, so that the quaternion treatment of differential geometry departs significantly from the formulation in terms of the Dirac algebra which we investigated in section 3.

An elementary spinor possesses one of four possible transformation laws, and each type of spinor has its own law of covariant differentiation. For convenience we list the four transformation laws with their appropriate covariant differentiation:

$$\left.\begin{array}{lll} \text{(i)} & \phi \to q\phi, & \phi_{;\mu} = \partial_\mu \phi - \Gamma_\mu \phi, \\ \text{(ii)} & \phi \to \phi q^{-1}, & \phi_{;\mu} = \partial_\mu \phi + \phi \Gamma_\mu, \\ \text{(iii)} & \phi \to \phi q^{\dagger}, & \phi_{;\mu} = \partial_\mu \phi - \phi \Gamma^{\dagger}_\mu, \\ \text{(iv)} & \phi \to q^{\dagger -1}\phi, & \phi_{;\mu} = \partial_\mu \phi + \Gamma^{\dagger}_\mu \phi, \end{array}\right\} \tag{5·11}$$

where Γ_μ is the elementary spinor connexion, with the transformation law

$$\Gamma_\mu \to q\Gamma_\mu q^{-1} + q_{.\mu} q^{-1} \, (q_{.\mu} = \partial_\mu q),$$

so that the matrices Γ_μ can be expressed as quaternions,

$$\Gamma_\mu = \beta_{\mu\nu} S^\nu,$$

where $\beta_{\mu\nu}$ is a tensor under coordinate transformations and, under Lorentz rotations of the vierbein,

$$\beta_{\mu\nu} \to \beta_{\mu\nu} + c_\nu q_{.\mu} q^{-1}.$$

The covariant derivatives of the σ_μ and S_μ are respectively

$$\sigma_{\mu;\rho} = D_\rho \sigma_\mu - \Gamma_\rho \sigma_\mu - \sigma_\mu \Gamma^{\dagger}_\rho$$

and

$$S_{\mu;\rho} = D_\rho S_\mu - \Gamma_\rho S_\mu + S_\mu \Gamma_\rho.$$

The second equation is unchanged if the Γ_ρ are modified by the addition of an arbitrary vector multiple of the unit matrix a_ρ, as was the case for the Dirac spinor connexion. The first equation, however, is unchanged only if a_ρ is *imaginary*.

The operator D_ρ implies differentiation of the σ_μ and S_μ as vectors, without regard to their spinor indices. Since the σ_a ($\sigma_\mu = h_\mu{}^a \sigma_a$) are the constant Pauli matrices in all frames, we have

$$D_\rho \sigma_\mu = \gamma_{\alpha\mu\rho} \sigma^\alpha,$$

where the $\gamma_{\alpha\mu\rho}$ are the same spin coefficients that were introduced for the Dirac algebra. We may also obtain $D_\rho \overline{\sigma}_\mu$, $D_\rho S_\mu$ and $D_\rho \overline{S}_\mu$ in terms of their components in the appropriate algebras. For instance,

$$D_\mu c = D_\mu (c_\alpha \sigma^\alpha) = (c_{\alpha;\mu} + c^\beta \gamma_{\alpha\beta\mu}) \sigma^\alpha$$

and

$$\begin{aligned} D_\mu \bar{c} &= -\bar{c}(D_\mu c)\bar{c} \\ &= -(c_{\alpha;\mu} + c^\beta \gamma_{\alpha\beta\mu}) \bar{c}\sigma^\alpha \bar{c}. \end{aligned}$$

But $\bar{c}\sigma^\alpha\bar{c} = -\overline{\sigma}^\alpha + 2\bar{c}c_\alpha$, and since c^α is a unit vector, $c^\alpha c_{\alpha;\mu} = 0$, so this reduces to

$$D_\mu \bar{c} = (c_{\alpha;\mu} + c^\beta \gamma_{\alpha\beta\mu}) \overline{\sigma}^\alpha.$$

Then

$$D_\mu S_\nu = D_\mu (\sigma_\nu \bar{c}) = S^\alpha \gamma_{\alpha\nu\mu} + \zeta_{\alpha\mu} \sigma_\nu \overline{\sigma}^\alpha,$$

where

$$\zeta_{\alpha\mu} = c_{\alpha;\mu} + c^\beta \gamma_{\alpha\beta\mu}.$$

Making use of the fact that $\sigma_\nu \bar{\sigma}_\alpha = S_\nu \bar{S}_\alpha$, and $\frac{1}{2}(S_\nu \bar{S}_\alpha + S_\alpha \bar{S}_\nu) = g_{\nu\alpha}$ we find that

$$D_\mu \bar{S}_\nu = -D_\mu S_\nu + c_{\nu;\mu}$$
$$= \bar{S}^\alpha \gamma_{\alpha\nu\mu} + \zeta_{\alpha\mu} \sigma^\alpha \bar{\sigma}^\nu,$$

and that

$$D_\mu \bar{\sigma}_\nu = D_\mu(\bar{c}\bar{S}_\nu) = \bar{\sigma}^\alpha \gamma_{\alpha\nu\mu} + 2c^\alpha \zeta_{\alpha\mu}$$
$$= \bar{\sigma}^\alpha \gamma_{\alpha\nu\mu}.$$

To obtain an expression for the spinor connexion Γ_μ we can set either $\sigma_{\nu;\mu} = 0$ or $S_{\nu;\mu} = 0$. The second choice, however, leads to the severe restriction $c_{\nu;\mu} = 0$ on the vector field c_ν, since $c_\nu S^\nu = 1$ implies $c_{\nu;\mu} S^\nu + c_\nu S^\nu{}_{;\mu} = 0$. We therefore apply the first condition

$$\sigma_{\nu;\mu} = 0$$

in the same way that $E_{\nu;\mu} = 0$ was used to specify the components of the Dirac spinor connexion. We have

$$\sigma_{\nu;\mu} = \sigma^\alpha \gamma_{\alpha\nu\mu} - \Gamma_\mu \sigma_\nu - \sigma_\nu \Gamma^\dagger_\mu = 0. \tag{5·12}$$

Writing

$$\Gamma_\mu = \beta_{\mu\alpha} S^\alpha$$

for the components of the Γ_μ, from (5·9),

$$\Gamma^\dagger_\mu = \beta^x_{\mu\alpha} \bar{c} S^\alpha \bar{c},$$

where x is complex conjugation. Equation (5·12) yields, on multiplication on the right by $\bar{\sigma}^\nu$,

$$0 = \sigma^\alpha \bar{\sigma}^\nu \gamma_{\alpha\nu\mu} - 4\Gamma_\mu - \beta^x_{\mu\alpha} \sigma_\nu \bar{c} S^\alpha c \bar{\sigma}^\nu$$
$$= S^{\alpha\nu} \gamma_{\alpha\nu\mu} - 4\beta_{\mu\alpha} S^\alpha - \beta^x_{\mu\alpha} S_\nu S^\alpha \bar{S}^\nu.$$

But $S_\nu S^\alpha \bar{S}^\nu = 4c^\alpha$, hence

$$0 = S^{\alpha\nu} \gamma_{\alpha\nu\mu} - 4\beta_{\mu\alpha} S^\alpha - 4\beta^x_{\mu\alpha} c^\alpha.$$

Taking the quaternionic conjugate and adding,

$$(\beta_{\mu\alpha} + \beta^x_{\mu\alpha}) c^\alpha = 0,$$

i.e. the quantity $\beta_{\mu\alpha} c^\alpha = a_\mu$ is imaginary, and the elementary spinor connexion is

$$\Gamma_\mu = \tfrac{1}{4} S^{\alpha\nu} \gamma_{\alpha\nu\mu} + a_\mu, \tag{5·13}$$

which bears a striking resemblance to the expression for the Dirac connexion in terms of E-numbers. From this treatment, however, we have shown that a_μ is necessarily imaginary. In the Dirac algebra treatment it was an arbitrary *complex* vector.

6. *Relationship between the Dirac and quaternion algebras.* There exists a close connexion between the Dirac and quaternion algebras, arising from the fact that the Dirac algebra is a direct product of two quaternion algebras. Thus every formulation of a mathematical or physical relationship in terms of E-numbers (for instance, the electromagnetic relations obtained in section 4) has a closely associated quaternion formulation of a similar type, that is equivalent. The Weyl representation of the Dirac algebra appears to be extremely appropriate and useful in this context, as

it provides a means of obtaining, from any E-number relation, an equivalent quaternion one. This fact is demonstrated in the present section and applied to the electromagnetic equations of section 4. In section 7 we then deal with what is in fact the central object of the paper—to show how the matrix methods evolved here may be used to simplify manipulations of tensor equations, in particular those tensor equations that arise from considerations of the gravitational field.

The Weyl representation of the Dirac algebra is given by the set of 4×4 matrices

$$E_\mu = \begin{pmatrix} & \sigma_\mu \\ \overline{\sigma}_\mu & \end{pmatrix} \quad (\mu = 0 \ldots 3). \tag{6·1}$$

We see immediately that the E_a are constants, that $E_{\mu;\nu} = 0$ follows from $\sigma_{\mu;\nu} = 0$ and that the four matrices satisfy

$$\tfrac{1}{2}(E_\mu E_\nu + E_\nu E_\mu) = g_{\mu\nu},$$

so that they do, in fact, generate a representation of the Dirac algebra. We find that the E^μ undergo the transformations

$$E_\mu \to QE_\mu Q^{-1}, \quad Q = \begin{pmatrix} q & \\ & \overline{q}^\dagger \end{pmatrix} \tag{6·2}$$

whenever the σ_μ undergo the Lorentz rotation q. E_5 is represented in all frames by

$$E_5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$

and we find

$$E_{\mu\nu} = \begin{pmatrix} \tfrac{1}{2}(\sigma_\mu \overline{\sigma}_\nu - \sigma_\nu \overline{\sigma}_\mu) & \\ & \tfrac{1}{2}(\overline{\sigma}_\mu \sigma_\nu - \overline{\sigma}_\nu \sigma_\mu) \end{pmatrix}$$

$$= \begin{pmatrix} S_{\mu\nu} & \\ & -S_{\mu\nu}{}^\dagger \end{pmatrix}, \tag{6·3}$$

where the † implies that the transformation law is $S_{\mu\nu}^\dagger \to \overline{q}^\dagger S_{\mu\nu}^\dagger q^\dagger$, although of course in the special frames the $S_{\mu\nu}$ are skewhermitian. Thus any E-number of the form $f = \tfrac{1}{2} f_{\mu\nu} E^{\mu\nu}$ is represented by

$$f = \begin{pmatrix} K & \\ & -K^\dagger \end{pmatrix}, \quad K = K_\mu S^\mu = \tfrac{1}{2} f_{\mu\nu} S^{\mu\nu},$$

which enables us to write down the electromagnetic equations (4·2) and the expression (4·3) for the energy-momentum tensor in terms of the quaternion K. The Maxwell equations are

$$f_{\mu\nu;\rho} S^\rho S^{\mu\nu} = f_{\mu\nu;\rho} S^{\nu\mu} S^\rho = j_\mu S^\mu,$$

and the expression

$$T_{\mu\nu} E^\nu = \tfrac{1}{2} f E_\mu f$$

gives

$$T_{\mu\nu} \begin{pmatrix} & \sigma^\nu \\ \overline{\sigma}^\nu & \end{pmatrix} = \frac{1}{2} \begin{pmatrix} K & \\ & -K^\dagger \end{pmatrix} \begin{pmatrix} & \sigma_\mu \\ \overline{\sigma}_\mu & \end{pmatrix} \begin{pmatrix} K & \\ & -K^\dagger \end{pmatrix} = \frac{1}{2} \begin{pmatrix} & -K\sigma_\mu K^\dagger \\ -K^\dagger \overline{\sigma}_\mu K & \end{pmatrix},$$

i.e.

$$T_{\mu\nu} \sigma^\nu = -\tfrac{1}{2} K \sigma_\mu K^\dagger. \tag{6·4}$$

Defining the 'complex conjugate' of a quaternion $K = K_\mu S^\mu$ to be the quantity

$$K^x = K_\mu^x S^\mu$$

(i.e. the basis elements of the algebra are regarded as real), then K^x has the same transformation law as K, and from (5·8) and (5·10) we see that

$$cK^{\dagger}\bar{c} = K^x.$$

Substituting in (6·4) we obtain
$$T_{\mu\nu}S^{\nu} = -\tfrac{1}{2}KS_{\mu}K^x.$$

7. *Gravitational theory in terms of the quaternion and Dirac algebras.* (*a*) *The Riemannian tensor and the curvature spinor.* It is convenient to introduce the Riemann tensor into the E-number formalism by demonstrating how it is involved in the spin-tensor $M_{\mu\nu}$ introduced in (3·7) in connexion with the commutator of two covariant derivatives of a Dirac spinor. A covariant Dirac spinor ψ transforms to $Q\psi$ under Lorentz rotations of the vierbein for which $E_{\mu} \rightarrow QE_{\mu}Q^{-1}$. Thus an E-number may be regarded as a rank two Dirac spinor with one covariant and one contravariant spinor index, and the covariant derivative of an E-number $a = a_{\mu}E_{\mu}$ is given by

$$\nabla_{\mu}a = \partial_{\mu}a - \Gamma_{\mu}a + a\Gamma_{\mu},$$

and the commutator of two such derivatives is a spin-tensor, given by

$$(\nabla_{\mu\nu} - \nabla_{\nu\mu})\, a = M_{\mu\nu}a - aM_{\mu\nu}.$$

On inserting the spinor indices the close connexion between this equation and the corresponding equation

$$(\nabla_{\mu\nu} - \nabla_{\nu\mu})\, a_{\alpha}{}^{\beta} = R_{\alpha}{}^{\rho}{}_{\mu\nu}a_{\rho}{}^{\beta} - a_{\alpha}{}^{\rho}R_{\rho}{}^{\beta}{}_{\mu\nu}$$

for a rank two tensor, and the consequent analogy between $M_{\mu\nu}$ and the Riemann tensor, becomes at once apparent. Since a_{μ} is a vector and $E_{\mu;\nu} = 0$ we can also formulate the commutator $\nabla_{[\mu\nu]}$ as follows:

$$\begin{aligned}(\nabla_{\mu\nu} - \nabla_{\nu\mu})\, a &= E^{\sigma}(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu})\, a_{\sigma}\\ &= E^{\sigma}R_{\sigma}{}^{\alpha}{}_{\mu\nu}a_{\alpha},\end{aligned}$$

and the right-hand side is seen from (2·10) to be

$$E^{\sigma}R_{\sigma}{}^{\alpha}{}_{\mu\nu}a_{\alpha} = \tfrac{1}{2}(K_{\mu\nu}a - aK_{\mu\nu}),$$

where
$$K_{\mu\nu} = \tfrac{1}{2}R_{\mu\nu\rho\sigma}E^{\rho\sigma}. \tag{7·1}$$

Thus
$$(M_{\mu\nu} - \tfrac{1}{2}K_{\mu\nu})\, a - a(M_{\mu\nu} - \tfrac{1}{2}K_{\mu\nu}) = 0$$

for any E-number a of the form $a_{\mu}E^{\mu}$. Hence $M_{\mu\nu} - \frac{1}{2}K_{\mu\nu}$ commutes with the E-number E_{μ} and so with all the E-numbers. It is therefore a multiple of the unit matrix. But $\mathrm{tr}\, K_{\mu\nu} = 0$ and $\mathrm{tr}\, M_{\mu\nu} = f_{\mu\nu}$, so we have

$$M_{\mu\nu} = \tfrac{1}{2}K_{\mu\nu} + f_{\mu\nu} = \tfrac{1}{4}R_{\mu\nu\rho\sigma}E^{\rho\sigma} + f_{\mu\nu}.$$

We shall use the E-numbers $K_{\mu\nu}$ and the associated quaternions

$$K_{\mu\nu} = \tfrac{1}{2}R_{\mu\nu\rho\sigma}S^{\rho\sigma}$$

to formulate gravitational expressions in terms of the Dirac and quaternion algebras, taking as a guide the methods we used to formulate electromagnetism in terms of f and K.

(*b*) *The Ricci and Einstein tensors.* The Ricci and Einstein tensors have very simple E-number expressions. Defining

$$R_\nu = K_{\mu\nu} E^\mu,$$

we find

$$R_\nu = \tfrac{1}{2} R_{\mu\nu\rho\sigma} E^{\rho\sigma\mu} = \tfrac{1}{2} R_{\mu\nu\rho\sigma}(g^{\mu\sigma} E^\rho - g^{\mu\rho} E^\sigma) = R_{\nu\mu} E^\mu.$$

The $E_{\mu 5}$ components vanish due to the cyclic identity. By virtue of the same identity

$$R_\nu = E^\mu K_{\nu\mu}.$$

The curvature scalar follows at once from

$$R_\nu E^\nu = R_{\mu\nu} E^\mu E^\nu = R_{\mu\nu} g^{\mu\nu} = R,$$

so that the Einstein tensor $G_{\mu\nu}$ is contained in

$$\begin{aligned} G_{\mu\nu} E^\nu &= (R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R) E^\nu = R_\mu - \tfrac{1}{2} R_\sigma E^\sigma E_\mu \\ &= R^\sigma (g_{\mu\sigma} - \tfrac{1}{2} E_\sigma E_\mu) \\ &= \tfrac{1}{2} R^\sigma E_\mu E_\sigma = \tfrac{1}{2} K^{\rho\sigma} E_{\rho\mu\sigma}. \end{aligned}$$

We thus have the following quantities which specify the Ricci and Einstein tensors in terms of E-numbers:

$$\left.\begin{aligned} R_{\mu\nu} E^\mu &= K_{\mu\nu} E^\mu, \\ G_{\mu\nu} E^\mu &= \tfrac{1}{2} K^{\rho\sigma} E_{\rho\nu\sigma}. \end{aligned}\right\} \tag{7·2}$$

The expression $K_{\mu\nu} E^\mu$ can be written

$$K_{\mu\nu} E^\mu = \tfrac{1}{2} E^{\alpha\beta} R_{\mu\nu\alpha\beta} E^\mu = \tfrac{1}{2} E^{\alpha\beta} D_{\alpha\beta} E_\nu,$$

which is essentially the expression for the Ricci tensor obtained by Newman and Kilmister (7).

Quaternion expressions for the Ricci and Einstein tensors. We can decompose equation (7·2) for the Ricci tensor into two (equivalent) quaternion expressions by using the Weyl representation of the E-numbers. In the Weyl representation,

$$K_{\mu\nu} = \begin{pmatrix} K_{\mu\nu} & \\ & -K_{\mu\nu}^{\dagger} \end{pmatrix},$$

so that

$$R_{\mu\nu} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} = \begin{pmatrix} K_{\mu\nu} & \\ & -K_{\mu\nu}^{\dagger} \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix},$$

which gives expressions for $R_{\mu\nu} \sigma^\mu$ and $R_{\mu\nu} \bar{\sigma}^\mu$, taking the quaternionic conjugate and the Hermitian conjugate of the second expression (and noting that $\sigma_\mu = \sigma_\mu^{\dagger}$, $\bar{K}_{\mu\nu} = -K_{\mu\nu}$), we find

$$\left.\begin{aligned} R_{\mu\nu} \sigma^\mu &= K_{\mu\nu} \sigma^\mu, \\ R_{\mu\nu} \sigma^\mu &= \sigma^\mu K_{\nu\mu}. \end{aligned}\right\}$$

The same two expressions follow from $R_{\mu\nu} E^\mu = E^\mu K_{\nu\mu}$.

To obtain the quaternion equivalents of the formula (7·2) for the Einstein tensor we expand $E_{\rho\nu\sigma}$ into its components thereby demonstrating that $G_{\mu\nu}$ is in fact the trace of the double dual of the Riemann tensor, just as $R_{\mu\nu}$ is the trace of the Riemann tensor itself. Thus

$$G_{\mu\nu}E^{\nu} = \tfrac{1}{2}K^{\rho\sigma}E_{\rho\mu\sigma} = \tfrac{1}{2}K^{\rho\sigma}(-e_{\rho\mu\sigma\lambda}E^{\lambda 5}),$$

the other components vanish due to the skewsymmetry in $(\rho\sigma)$. Therefore

$$G_{\mu\nu}E^{\nu} = ({}^*K_{\mu\lambda})\,E^{\lambda 5},$$

where ${}^*K_{\mu\lambda} = {}^*R_{\mu\lambda\alpha\beta}E^{\alpha\beta}$, the star on the left of the Riemann tensor indicating that the dual has been taken over the left-hand index pair $(\mu\lambda)$. Similarly $R^*{}_{\mu\lambda\alpha\beta}$ indicates a dual taken over $(\alpha\beta)$.

$${}^*K^*{}_{\mu\lambda} = \tfrac{1}{2}{}^*R^*{}_{\mu\lambda\alpha\beta}E^{\alpha\beta} = \tfrac{1}{2}{}^*R_{\mu\lambda\alpha\beta}E^{\alpha\beta 5} = {}^*K_{\mu\lambda}E^{5},$$

therefore

$$G_{\mu\nu}E^{\nu} = {}^*K^*{}_{\lambda\mu}E^{\lambda}. \tag{7·3}$$

Comparing this expression with the analogous expression for the Ricci tensor (7·2), and noting that the double dual of the Riemann tensor has the same algebraic symmetries as the Riemann tensor itself, we have shown that the relationship between the Einstein tensor and the double dual of the Riemann tensor is the same as that between the Ricci tensor and the Riemann tensor. $G_{\mu\nu}$ is therefore the trace of ${}^*R^*{}_{\rho\mu\nu\sigma}$.

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}G_{\mu\nu}R = {}^*R^*{}_{\rho\mu\nu}{}^{\rho}. \tag{7·4}$$

The quaternion equivalent of (7·2) follows immediately:

$$G_{\mu\nu}\sigma^{\mu} = {}^*K^*{}_{\lambda\mu}\sigma^{\lambda},$$

where ${}^*K^*{}_{\lambda\mu} = \tfrac{1}{2}{}^*R^*{}_{\lambda\mu\alpha\beta}S^{\alpha\beta}$. Since $S^{\alpha\beta}$ is self dual the right-hand star can be omitted. Thus

$$G_{\mu\nu}S^{\mu} = {}^*K_{\lambda\mu}S^{\lambda}. \tag{7·5}$$

(c) *Analogy between $K_{\mu\nu}$ and $f_{\mu\nu}$.* The quantity $K_{\mu\nu}$ appears to be analogous in some ways to the electromagnetic quantity $f_{\mu\nu}$. If the analogy were complete we would expect the gravitational equations to be the Maxwell equations (4·1) with $K_{\mu\nu}$ in place of $f_{\mu\nu}$. The equations

$$\left.\begin{aligned} SK_{\mu\rho;\sigma} &= 0,\\ K^{\rho\mu}{}_{;\rho} &= J^{\mu}\end{aligned}\right\} \tag{7·6}$$

have in fact been put forward at various times as a possible alternative gravitational theory to the one based on the Einstein equations. The semicolon denotes complete derivative. See, for example, Lichnerowicz (8), Kilmister and Newman (7) and Bel (1). The first equation is an identity—the Bianchi identity. The second is a set of third-order restrictions on the metric, which in empty space ($J^{\mu} = 0$) include Einstein's equations. That is, $K^{\rho\mu}{}_{;\rho} = 0$ whenever $R_{\mu\nu} = 0$. The quantity J^{μ}, unlike the electric current j^{μ}, is not a multiple of the unit matrix but is an E-number

$$J^{\mu} = J^{\mu\rho\sigma}E_{\rho\sigma} = \tfrac{1}{2}R_{\rho\sigma}{}^{\nu\mu}{}_{;\nu}E^{\rho\sigma}. \tag{7·7}$$

Making use of the Bianchi identity gives the alternative expression

$$J^{\mu} = \tfrac{1}{2}(R^{\mu}{}_{\rho;\sigma} - R^{\mu}{}_{\sigma;\rho})\,E^{\rho\sigma}.$$

We have

$$\begin{aligned}\nabla_\mu J^\mu &= \tfrac{1}{2}\nabla_{\mu\nu} R_{\rho\sigma}{}^{\nu\mu} E^{\rho\sigma} \\ &= \tfrac{1}{4}(\nabla_{\mu\nu} - \nabla_{\mu\nu}) R^{\rho\sigma\nu\mu} E_{\rho\sigma} \\ &= \tfrac{1}{4}(R^{\alpha\sigma\nu\mu} R^\rho{}_{\alpha\nu\mu} + R^{\rho\alpha\nu\mu} R^\sigma{}_{\alpha\nu\mu} - R^{\sigma\alpha\mu} R_{\alpha\mu} + R^{\rho\sigma\nu\alpha} R_{\alpha\nu}) E_{\rho\sigma}.\end{aligned}$$

The first two terms in brackets are symmetric in $(\rho\sigma)$, while the second two vanish because the Riemann tensor is skew in its final index pair, while the Ricci tensor is symmetric. Hence J^μ obeys a continuity equation

$$\nabla_\mu J^\mu = 0,$$

or

$$\nabla_\mu J^{\mu\rho\sigma} = 0. \tag{7·8}$$

The cyclic property of $R^{\mu\nu\rho\sigma}$ implies also a similar property on J^μ;

$$SJ_{\mu\rho\sigma} = 0.$$

Combining this with the continuity equation (7·8) and the skewsymmetry in $(\rho\sigma)$ gives the continuity equation

$$\nabla_\mu I^\mu = 0 \quad \text{where} \quad I^\mu = J^{\rho\sigma\mu} E_{\rho\sigma}.$$

8. *The EF-number algebra.* (*a*) *Ricci and Einstein tensors.* The E-numbers may be regarded as formed from two mutually commuting quaternion algebras, or as linear functions of quaternions. In a similar way we can construct an algebra from two mutually commuting Dirac algebras, which is a modified form of C_8. We take a set of quantities E_μ, F_μ, with the following multiplication rules

$$\begin{aligned}\tfrac{1}{2}(E^\mu E^\nu + E^\nu E^\mu) &= g^{\mu\nu} = \tfrac{1}{2}(F^\mu F^\nu + F^\nu F^\mu), \\ E^\mu E^\nu &= E^\nu F^\mu.\end{aligned}$$

A representation can be constructed from 8×8 matrices by Kronecker products on a 4×4 representation of the E-numbers. Thus

$$E^\mu = (E^\mu \otimes 1), \quad F^\mu = (1 \otimes E^\mu).$$

The identification of components of Clifford algebras with physical quantities is now less restricted, since the $E^\mu F^\nu$ component of a general EF-number is a tensor of rank two $P_{\mu\nu}$, we are no longer restricted to *skewsymmetric* rank two tensor as we were with the single E-number algebra. The algebraic properties of an E-number of the form

$$P = \tfrac{1}{4} P_{\mu\nu\rho\sigma} E^{\mu\nu} F^{\rho\sigma}$$

are of particular importance, for the tensor $P_{\mu\nu\rho\sigma}$ is skewsymmetric in its index pairs $(\mu\nu)$ and $(\rho\sigma)$. If it is also symmetric under interchange of index pairs this fact is equivalent to the invariance of P under the *interchange* of the roles of E_μ and F_μ. We form the quantity

$$K = \tfrac{1}{4} R_{\mu\nu\rho\sigma} E^{\mu\nu} F^{\rho\sigma} \tag{8·1}$$

from the Riemann tensor, and also define

$$I = g_{\mu\nu} E^\mu F^\nu = E_\mu F^\mu$$

from the metric tensor. Then

$$
\begin{aligned}
IK &= \tfrac{1}{4}R_{\mu\nu}{}^{\rho\sigma}(E^{\alpha\mu\nu}F_{\alpha\rho\sigma}) \\
&= \tfrac{1}{4}R_{\mu\nu}{}^{\rho\sigma}(e^{\lambda\alpha\mu\nu}E_{\lambda 5}+2g^{\alpha\mu}E^{\nu})(e_{\kappa\alpha\rho\sigma}F^{\kappa 5}+2g_{\alpha\rho}F_{\sigma}) \\
&= \tfrac{1}{4}R_{\mu\nu}{}^{\rho\sigma}(-3!\,\delta^{\lambda\mu\nu}_{[\kappa\rho\sigma]}E_{\lambda 5}F^{\kappa 5})-R_{\nu\sigma}E^{\nu}F^{\sigma},
\end{aligned}
$$

where the terms in $R_{\mu}{}^{\nu\rho\sigma}e_{\kappa\nu\rho\sigma}$, etc., vanish by virtue of the cyclic identity. Now,

$$3!\,\delta^{\lambda\mu\nu}_{[\kappa\rho\sigma]}R_{\mu\nu}{}^{\rho\sigma} = -2\delta_{\kappa}{}^{\lambda}R+4R_{\kappa}{}^{\lambda},$$

so we have

$$IK = -(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)\,E^{\mu 5}F^{\nu 5}-R_{\mu\nu}E^{\mu}F^{\nu},$$

and we also note that $IK \equiv KI$ is a consequence of the cyclic identity. If we denote by $*K*$ the quantity analogous to K formed from the double dual of the Riemann tensor, we have

$$*K* = E_5F_5K = KE_5F_5,$$

so

$$
\left.\begin{aligned}
IK &= -G_{\mu\nu}E^{\mu 5}F^{\nu 5}-R_{\mu\nu}E^{\mu}F^{\nu}, \\
I(*K*) &= -R_{\mu\nu}E^{\mu 5}F^{\nu 5}-G_{\mu\nu}E^{\mu}F^{\nu}.
\end{aligned}\right\} \qquad (8\cdot 2)
$$

We note that the roles of the Ricci and Einstein tensors are interchanged when $R_{\mu\nu\rho\sigma}$ is replaced by its double dual. This fact was indicated in the treatment of the Einstein tensor in the previous section, but here in the context of EF-numbers it finds its full expression as a consequence of the algebraic properties.

(*b*) *Bel's tensor.* Bel (1) introduced the tensor

$$T_{\mu\rho\nu\sigma} = \tfrac{1}{2}(R_{\mu\alpha\nu\beta}R_{\rho}{}^{\alpha}{}_{\sigma}{}^{\beta}-{*R}_{\mu\alpha\nu\beta}{*R}_{\rho}{}^{\alpha}{}_{\sigma}{}^{\beta}-R^{*}{}_{\mu\alpha\nu\beta}R^{*}{}_{\rho}{}^{\alpha}{}_{\sigma}{}^{\beta}+{*R^{*}}{}_{\mu\alpha\nu\beta}{*R^{*}}{}_{\rho}{}^{\alpha}{}_{\sigma}{}^{\beta}) \qquad (8\cdot 3)$$

in connexion with the energy density of the gravitational field. The algebraic properties of such a tensor are tedious and complicated to deal with if the only mathematical methods used are the usual manipulation of tensor indices. We show how the E-number method of dealing with the electromagnetic energy-momentum tensor can be extended to deal with a rank four tensor by using EF-numbers. The tensor that we would expect by direct analogy with (4·6) is in fact Bel's tensor, and the proof of the various symmetry properties is greatly simplified.

By analogy with (4·3) we are led to consider the EF-number

$$
\begin{aligned}
KE^{\mu}F^{\nu}K &= \tfrac{1}{16}R_{\alpha\beta\gamma\delta}R_{\kappa\lambda\rho\sigma}(E^{\alpha\beta\mu\kappa\lambda})(F^{\gamma\delta\nu\rho\sigma}) \\
&= \tfrac{1}{16}R_{\alpha\beta\gamma\delta}R_{\kappa\lambda\rho\sigma}\{(e^{\alpha\beta\mu a}e_{a}{}^{\kappa\lambda b}+4g^{\beta\mu}g^{\alpha\kappa}g^{\lambda b})\,E_{b} \\
&\qquad +(2e^{\alpha\beta\mu\kappa}g^{\lambda b}+2g^{\beta\mu}e^{\alpha\kappa\lambda b})\,E_{b5}\}\,(F^{\gamma\delta\nu\rho\sigma}) \\
&= \tfrac{1}{4}(R_{\alpha\mu\gamma\delta}R^{\alpha\lambda}{}_{\rho\sigma}-{*R}_{\alpha\mu\gamma\delta}{*R}^{\alpha\lambda}{}_{\rho\sigma})\,E_{\lambda}(F^{\gamma\delta\nu\rho\sigma}) \\
&\qquad +\tfrac{1}{4}(R_{\alpha\mu\gamma\delta}{*R}^{\alpha\lambda}{}_{\rho\sigma}-{*R}_{\alpha\mu\gamma\delta}R^{\alpha\lambda}{}_{\rho\sigma})\,E_{\lambda 5}\,(F^{\gamma\delta\nu\rho\sigma}),
\end{aligned}
$$

so that, if the $F^{\gamma\delta\nu\rho\sigma}$ were also expanded by components, we see that $KE^{\mu}F^{\nu}K$ would contain only $E_{\lambda}F_{\kappa}$, $E_{\lambda 5}F_{\kappa}$, $E_{\lambda}F_{\mu 5}$ and $E_{\lambda 5}F_{\kappa 5}$ components, the component associated with $E_{\lambda}F_{\kappa}$ being the $2T_{\mu\lambda\nu\kappa}$ of (8·3). The trace of an EF-number in a 16-dimensional representation is obtained by regarding the E_{μ} as numbers and taking the trace as an F-number, and then taking the trace of the resulting 4×4 matrix (E-number). We can write $\operatorname{tr}P = \operatorname{tr}_E(\operatorname{tr}_F P)$. With this convention, we have the following result:

$$T_{\mu\lambda\nu\kappa} = \tfrac{1}{32}\operatorname{tr}E_{\lambda}F_{\kappa}KE_{\mu}F_{\nu}K. \qquad (8\cdot 4)$$

The symmetry under interchange of index pairs, $(\lambda\kappa)$ with $(\mu\nu)$, follows immediately by cyclically permuting the matrices on which the trace operates. We may also interchange the E-numbers with the F-numbers. Under this operation K is invariant so the net effect is to interchange the index pairs $(\lambda\mu)$ with $(\kappa\nu)$. This establishes the following two symmetries of Bel's tensor:

$$T_{\mu\lambda\nu\kappa} = T_{\lambda\mu\kappa\nu}, \tag{8·5}$$

$$T_{\mu\lambda\nu\kappa} = T_{\nu\kappa\mu\lambda}. \tag{8·6}$$

Proof of the symmetry in $(\nu\kappa)$ is as follows:

$$\begin{aligned} T_{\mu\lambda\nu\kappa} &= \tfrac{1}{32}\cdot\tfrac{1}{16} R^{\alpha\beta\gamma\delta} R^{abcd} \operatorname{tr}(E_{\mu\alpha\beta\lambda ab} F_{\nu\gamma\delta\kappa cd}) \operatorname{tr}(E_{\mu\alpha\beta\lambda ab} F_{\nu\gamma\delta\kappa cd}) \\ &= \operatorname{tr}_E(E_{\mu\alpha\beta\lambda ab}) \operatorname{tr}_F(F_{\nu\gamma\delta\kappa cd}), \end{aligned}$$

and the part of $F_{\nu\gamma\delta\kappa cd}$ that does not vanish when multiplied by $R^{\alpha\beta\gamma\delta}R^{abcd}$ is

$$(e_{\nu\gamma\delta\rho}F^{\rho 5} + 2g_{\nu\gamma}F_\delta)(e_{\kappa c d\sigma}F^{\sigma 5} + 2g_{\kappa c}F_d),$$

and the part of this quantity with non-vanishing trace is

$$-e_{\nu\gamma\delta\rho}e_{\kappa cd}{}^{\rho} + 4g_{\nu\gamma}g_{\kappa c}g_{\delta d},$$

which, when multiplied by a quantity skew in $(\gamma\delta)$ and in (cd) is equivalent to

$$-2g_{\nu\kappa}g_{\gamma c}g_{\delta d} + 4g_{\kappa\gamma}g_{\nu c}g_{\delta d} + 4g_{\nu\gamma}g_{\kappa c}g_{\delta d}$$

which is symmetric in $(\nu\kappa)$. This establishes the symmetry

$$T_{\mu\lambda\nu\kappa} = T_{\mu\lambda\kappa\nu}. \tag{8·7}$$

The remaining symmetry property, symmetry in $(\mu\nu)$, will, together with (8·5), (8·6) and (8·7), be sufficient to establish the complete symmetry of Bel's tensor. This symmetry is, however, only valid if the Riemann tensor satisfies the Einstein equations. The proof follows. Let $P_{\mu\rho\nu\sigma}$ be the $E^{\rho 5}F^{\sigma}$ component of $\tfrac{1}{2}KE_\mu F_\nu K$. Then, treating $\tfrac{1}{2}KE_\mu F_\nu K$ as an F-number, the E_μ being regarded as (non-commuting) numbers, we have

$$\tfrac{1}{8}\operatorname{tr}_F(F_\sigma K E_\mu F_\nu K) = T_{\mu\rho\nu\sigma}E^{\rho} + P_{\mu\rho\nu\sigma}E^{\rho 5}.$$

Now $$\tfrac{1}{8}E^{\sigma}\operatorname{tr}_F(F_\sigma K E_\mu F_\nu K) = \tfrac{1}{32}\operatorname{tr}_F(E^{\sigma}F_\sigma K E_\mu F_\nu K)$$

vanishes if Einstein's empty-space equations hold, since then $E^{\sigma}F_{\sigma} K = IK = 0$ by (8·2). Hence

$$T_{\mu\rho\nu\sigma}E^{\sigma}F^{\rho} + P_{\mu\rho\nu\sigma}E^{\sigma}F^{\rho 5} = 0. \tag{8·8}$$

Similarly, since $\tfrac{1}{32}\operatorname{tr}_F(KE_\mu F_\nu K F_\sigma E^{\sigma}) = 0$ we have

$$T_{\mu\rho\nu\sigma}E^{\rho}F^{\sigma} + P_{\mu\rho\nu\sigma}E^{\rho 5}F^{\sigma} = 0. \tag{8·9}$$

Subtracting (8·9) from (8·8)

$$T_{\mu\rho\nu\sigma}E^{\sigma\rho} + P_{\mu\rho\nu\sigma}g^{\rho\sigma}E^{5} = 0$$

and the $E^{\sigma\rho}$ component of this equation gives the required symmetry

$$T_{\mu\rho\nu\sigma} = T_{\mu\sigma\nu\rho}. \tag{8·10}$$

The vanishing of the trace $T_{\mu\rho\nu}{}^{\rho}$ when Einstein's equations are obeyed follows from adding (8·8) and (8·9), or from noting that

$$T_{\mu\rho\nu}{}^{\rho} = \tfrac{1}{32}\,\mathrm{tr}\,(E_{\rho}F^{\rho}KE_{\mu}F_{\nu}K) = 0.$$

The other trace, of the form $T_{\mu}{}^{\mu}{}_{\nu\rho} = T_{\nu\rho\mu}{}^{\mu}$ vanishes identically, due to the identity $E_{\mu}KE^{\mu} = 0$. We note that the symmetries (8·5), (8·6), (8·7) and (8·10) are sufficient to establish the *complete symmetry* of Bel's tensor, whenever the Riemann tensor satisfies the Einstein gravitational equations.

(*c*) *Divergence of Bel's tensor.* Finally, we may calculate an expression for the divergence $\nabla_{\mu}T^{\mu\rho\nu\sigma}$ of Bel's tensor. We define the EF-number

$$J = J_{\alpha\beta\gamma}E^{\alpha}F^{\beta\gamma}$$

from the tensor $J_{\alpha\beta\gamma}$ introduced in (7·7). We find that

$$\begin{aligned}\nabla K &= \tfrac{1}{4}R_{\alpha\beta\gamma\delta;\mu}E^{\mu\alpha\beta}F^{\gamma\delta}\\ &= \tfrac{1}{2}E_{\alpha\beta\gamma\delta;\mu}g^{\mu\alpha}E^{\beta}F^{\gamma\delta} = -J,\end{aligned}$$

where the term in $E_{\lambda 5}$ vanishes due to the Bianchi identity. Similarly $K\nabla = J$, so that

$$\begin{aligned}\nabla_{\mu}T^{\mu\rho\nu\sigma} &= \tfrac{1}{32}\nabla_{\mu}\,\mathrm{tr}\,(E^{\mu}F^{\nu}KE^{\rho}F^{\sigma}K)\\ &= \tfrac{1}{32}\,\mathrm{tr}\,(E^{\mu}F^{\nu}\nabla_{\mu}KE^{\rho}F^{\sigma}K + E^{\mu}F^{\nu}KE^{\rho}F^{\sigma}\nabla_{\mu}K)\\ &= \tfrac{1}{32}\,\mathrm{tr}\,(JF^{\nu}KE^{\rho}F^{\sigma}K - JE^{\rho}F^{\sigma}KF^{\nu}).\end{aligned}$$

This quantity vanishes if Einstein's equations are obeyed, since $J = R_{\mu\alpha;\beta}E^{\mu}F^{\alpha\beta}$. We calculate

$$\begin{aligned}F_{\nu}K &= \tfrac{1}{4}R^{\alpha\beta\gamma\delta}E_{\alpha\beta}F_{\nu\gamma\delta}\\ &= \tfrac{1}{2}R^{*\alpha\beta\nu\lambda}E_{\alpha\beta}F_{\lambda 5} + \tfrac{1}{2}R^{\alpha\beta\nu\lambda}E_{\alpha\beta}F_{\lambda},\end{aligned}$$

and similarly, $$KF^{\nu} = \tfrac{1}{2}R^{*\alpha\beta\nu\lambda}E_{\alpha\beta}F_{\lambda 5} - \tfrac{1}{2}R^{\alpha\beta\nu\lambda}E_{\alpha\beta}F_{\lambda}.$$

Hence $$\begin{aligned}F^{\nu}KE_{\rho}F_{\sigma} - E_{\rho}F_{\sigma}KF^{\nu} &= \tfrac{1}{2}R^{*\alpha\beta\nu\lambda}(E_{\alpha\beta\rho}F_{\lambda 5\sigma} - E_{\rho\alpha\beta}F_{\sigma}F_{\lambda})\\ &= 2\begin{Bmatrix} -{}^{*}R^{*}{}_{\rho}{}^{\gamma\nu\lambda}E_{\gamma 5}g_{\lambda\sigma}F_{5}\\ +{}^{*}R_{\rho}{}^{\gamma\nu\lambda}E_{\gamma 5}g_{\sigma\lambda}\\ -R^{*}{}_{\rho}{}^{\gamma\nu\lambda}E_{\gamma}F_{\sigma\lambda}{}^{*}\\ +R_{\rho}{}^{\gamma\nu\lambda}E_{\gamma}F_{\sigma\lambda}\end{Bmatrix}.\end{aligned}$$

So that, on multiplying by $J = J_{abc}E^{a}F^{bc}$, the only terms with non-vanishing trace that we obtain are

$$4(J^{*}{}_{a\sigma c}R^{*a\rho c\nu} - J_{a\sigma c}R^{a\rho c\nu}),$$

so that we have established the result that

$$\nabla_{\mu}T^{\mu\rho\nu}{}_{\sigma} = \tfrac{1}{8}(R^{*a\rho c\nu}J^{*}{}_{a\sigma c} - R^{a\rho c\nu}J_{a\sigma c}).$$

We note that on contraction over $(\nu\sigma)$, the duals on the Riemann tensor and on the final index pair of $J_{a\sigma c}$ may be dropped, so that $\nabla_{\mu}T^{\mu\rho\nu}{}_{\nu} = 0$, which is to be expected since we have already shown that $T^{\mu\rho\nu}{}_{\nu}$ is identically zero. Contraction over $(\rho\nu)$ gives

$$\nabla_{\mu}T^{\mu\rho}{}_{\rho\sigma} = R^{\alpha\beta}J_{\alpha\sigma\beta}.$$

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